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A GENERALIZED PAYOFF FOR DIFFERENTIAL GAME ROLE DETERMINATION IN A TWO AIRCRAFT COMBAT

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relative advantage.

The class of game is two person, free time, zero sum and perfect information. The aircraft dynamics are restricted to constant speed in the horizontal plane. The controls for both players are limited turn rates. A fixed weapon range establishes a terminal constraint around one player. Using limits on the payoff, a game of kind is formulated to give successful termination at weapon range for both players. The payoff is based on angular terminal requirements, and allows for a variety of weapons. Closed form solutions are used to find the solution in the large by constructing trajectories and surfaces backwards from cost criteria on the terminal surface.

A complete solution is not obtained, but the existence of isolated singular arcs for this type of game is confirmed, and progress is made towards fully partitioning the game space. Many resulting trajectories correspond to realistic scissors maneuvers, and a method of partial role determination is proposed, using such trajectories to indicate regions of advantage.

A GENERALIZED PAYOFF FOR DIFFERENTIAL

GAME ROLE DETERMINATION IN A

TWO AIRCRAFT COMBAT

Thesis

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III

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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

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Preface

The application of differential game theory to the analysis of aerial combat offers the fascinating prospect of evaluating weapon systems and combat tactics against an intelligent, evasive opponent. Unfortunately, it seems the more complex the model, the harder it is to maintain validity. Consequently differential gaming has not yet made a substantial impact on realistic combat analysis.

One of the limiting factors has been that each player has had fixed roles of either pursuer or evader. This work is the result of my efforts to evaluate a method which allows for a change of roles. The study is based on a general purpose payoff first suggested in the Doctoral Dissertation of Major Urban H. D. Lynch, then at the Air Force Flight Dynamics Laboratory. Although both the game model and the constant speed, horizontal dynamic model are really too simple, a particularly fascinating result was trajectories that correspond to realistic scissors maneuvers.

I am very grateful to my advisor, Professor Gerald M. Anderson of the Air Force Institute of Technology, whose knowledge and advice were invaluable. I am also grateful to many of my fellow students, in particular Captain Robert D. Powell, without whose practical experience of the problem, I would have been fishing in murkier waters.

P. G. Jenkins

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Abbreviations

A	Player representing aircraft on which co-ordinate system is located
B	Other aircraft
BUP	Boundary of the Usable Part
DS	Dispersal Surface
TS	Terminal Surface
w.r.t.	With respect to

Symbols

a	Terminal weighting factor in Lynch payoff
b	Terminal weighting factor in Lynch payoff
c	Time integral weighting factor in Lynch payoff
G	Game space
H	Hamiltonian
H_o	Part of H which is a function of the states
H_A	Part of H which is a function of player A's controls
H_B	Part of H which is a function of player B's controls
J	Cost function
J_A	Limit of J defining A's successful outcome
J_B	Limit on J defining B's successful outcome
J_{min}	Minimum value of J
J_{max}	Maximum value of J
L	Weapon Range

R	Region of G resulting in a draw in optimal play
R_{SA}	Region of G resulting in success for A
R_{SB}	Region of G resulting in success for B
R_{SAB}	Region of G resulting in a mutual kill
R_{AA}	Region of G in which A has an advantage
R_{AB}	Region of G in which B has an advantage
R_A	Minimum radius of turn for A
R_B	Minimum radius of turn for B
r	Distance between A and B
r_f	Distance between A and B at the final time
S_A	Switching function for player A
S_B	Switching function for player B
t	Independent time variable
t_o	Initial time
t_f	Final time
t_s	Switching time at the end of an arc
U_A	A's control or turn rate
U_B	B's control or turn rate
U_{AMAX}	A's maximum turn rate
U_{BMAX}	B's maximum turn rate
V_A	Velocity of A
V_B	Velocity of B
v	Speed ratio of A w.r.t B
$\underline{x}$	Vector of the state variables
$\underline{x}_B$	Point on a B singular arc
$\underline{x}_{DS}$	Dispersal point

x, y	Rotating cartesian co-ordinate system giving the position of B relative to A
x_f, y_f	x and y at the final time
x_s, y_s	x and y at a switching time
z	State representing the angle between B's velocity and A's velocity vector
z_f	z at the final time
z_s	z at a switching time
θ	Angle between A's velocity vector and B's position vector
θ_A	Minus θ
θ_B	Angle between B's position vector and B's velocity vector
θ_f	θ at the final time
$\underline{\lambda}$	Co-state vector
$\lambda_r, \lambda_\theta, \lambda_{r_f}, \lambda_{\theta_f}$	Co-states for r, θ; and at the final time
$\lambda_{x_f}, \lambda_{y_f}$	Co-states for x, y at the final time
$\lambda_{x_s}, \lambda_{y_s}$	Co-states for x, y at a switching time
λ_z	Co-state for z
λ_{z_f}	Co-state for z at the final time
λ_{z_s}	Co-state for z at a switching time
ν	Constant multiplier
σ_A, σ_B	A's control on an arc, B's control on an arc
$\sigma_{A_f}, \sigma_{B_f}$	A's control at the final time, B's control at the final time
σ_r	Ratio of B's control to A's control
ϕ	Terminal payoff
ϕ_A, ϕ_B	Limits on payoff for A's success, and B's success

ϕ'_A, ϕ'_B	Expanded limits on the payoff
ϕ_N	Payoff representing a neutral outcome
ϕ_{min}	Minimum payoff on the terminal surface
ϕ_z	Payoff at point $\underline{Z}$ on the terminal surface
ψ	Terminal constraint
ψ_A, ψ_B	Terminal constraint for A, and terminal constraint for B
ψ_D	Terminal constraint for a draw outcome

Abstract

The problem investigated is a method of role determination, using a general purpose payoff, in a differential game model of aerial combat. Role determination is taken to mean a combatant's selection of a combat objective, based on his relative advantage.

The class of game is two person, free time, zero sum and perfect information. The aircraft dynamics are restricted to constant speed in the horizontal plane. The controls for both players are limited turn rates. A fixed weapon range establishes a terminal constraint around one player. Using limits on the payoff, a game of kind is formulated to give successful termination at weapon range for both players. The payoff is based on angular terminal requirements, and allows for a variety of weapons. Closed form solutions are used to find the solution in the large by constructing trajectories and surfaces backwards from cost criteria on the terminal surface.

A complete solution is not obtained, but the existence of isolated singular arcs for this type of game is confirmed, and progress is made towards fully partitioning the game space. Many resulting trajectories correspond to realistic scissors maneuvers, and a method of partial role determination is proposed, using such trajectories to indicate regions of advantage.

I. Introduction

Experience over the last decade has re-emphasized the importance of aerial combat in air warfare. This has resulted in a number of different efforts to improve the design and operation of combat aircraft and their weapons by realistically modelling the combat situation.

The energy maneuverability approach is useful for reaching and maintaining a generally advantageous position, though with no guidance as to what to do with that position (Ref 4). More extensive optimisation techniques overcome this drawback to some extent, but are limited to a restricted target model (Ref 2). Computer simulations have been designed to combine energy considerations with certain established guidance laws and role logic (Ref 12). Using this method realistic encounters result, and useful trade off studies are possible, though they are limited by the set of simulation rules used. Manual combat simulators are powerful tools, combining "seat of the pants" and scientific analysis; they are, however, highly subjective. At completely the opposite end of the spectrum is gaming, and in particular, continuous (differential) gaming (Ref 9).

The potential advantage of gaming is the introduction of active opponents into a completely analytical model. However, in application to realistic problems, success has been limited. One limitation has been that many applications have used a pursuit-evasion game, where one player is a pursuer, and the other is an evader. The pursuer's aim is to destroy

the evader, whose aim is to escape. The roles of each player are fixed throughout the game. Realistically however, in combat between relatively similar aircraft and weapon systems, <u>each</u> wishes to destroy his opponent. Allowance must be made for this in a game analysis.

The purpose of this investigation is to allow for a change of roles by using a general purpose payoff. The basic game model is zero sum, two person, free time, and perfect information. The dynamics are restricted to a planar, constant velocity model, in which each player has a limited turn capability.

The problem is discussed in Chapter II. The origins and nature of the problem are related to general ideas of role determination, and the thesis objective is stated. Game and dynamic models are established, with accompanying assumptions and constraints. Finally the payoff is discussed, and general criteria for determining the outcome of the game are set up.

Chapter III formulates the game fully: the payoff is modified to the game, and the necessary conditions then applied to the problem.

In Chapter IV closed form solutions to the game are derived for arcs of constant control, and for game surfaces located at arc junctions.

In Chapter V the closed form solutions are used to find a complete solution in the large.

Chapter VI discusses the role determination problem as

general, and in relation to the results. Some comparison with other game models is made.

It is felt this study makes a definite contribution to the differential game modelling of aerial combat. The use of the Lynch payoff in the game results in closed form solutions, which generate trajectories corresponding to practical scissors maneuvering. Although complete analysis awaits a full solution, significant extensions are made to the knowledge about this type of game and payoff. Finally, a method is proposed for determining regions of advantage and disadvantage to either combatant.

II. Discussion of the Problem

Origins of the Problem

The problem considered here is an extension of work carried out at the USAF Flight Dynamics Laboratory in the use of differential gaming to model air-to-air combat. Specifically, it is a continuation of work on a general purpose payoff begun by Lynch in his Doctoral Dissertation (Ref 9:174).

Nature of the Problem

A serious limitation in many of the results of Reference 9 is that the aim (objective) of each aircraft (player) is fixed throughout the game; usually one aircraft is trying to escape, and the other to capture. Some useful indications of the form and sensitivity of escape and capture regions have resulted from this work. However, the optimal control sequences (strategies) produced, sometimes differ considerably from those expected from experience, thus questioning the validity of that approach. For example, the slower machine, unless considerably more maneuverable than its opponent, will always, under optimal play, be captured. Thus the slow evader will invariably turn away from the pursuer until, at some stage, he flies directly away, and is then caught directly from the rear (Ref 9:80)! In actual combat, the objectives may change. Assuming the performance of each aircraft and its weapon system is in roughly the same class, then usually

each combatant is, at some stage, hoping for success. Thus the strategies are not necessarily ones of pure pursuit or escape and should reflect an aggressiveness by each combatant.

Based on Baron's approach to this "dogfight" situation (Ref 3:65), the problem of aerial combat is considered to consist of two main parts:

1. Given the states of two players, determine the role of each. Here role determination is taken to mean:
    a. Which player has an advantage, and to what extent?
    b. Based on this relative advantage, what is the objective of each player?
2. Given the roles, what are the optimal strategies for each player?

Of the two major parts, role determination is the unique and vital aspect of the combat problem. Other work has been done on role determination (Refs 10, 11); the important difference here is that the payoff emphasizes the relative angles between the combatants at termination.

Thesis Objectives

The primary objective is to study the validity of using the general purpose payoff in role determination. The secondary objective, although to a large extent inseparable from the first, is to develop a technique for using this payoff to analyze fully simple problems, and to form a base for the addition of more realistic aircraft dynamics.

Basic Game Model

The combat is modeled as a two person (aircraft), free time, zero sum, perfect information game. A zero sum game implies pure competition, and although combat games are generally regarded as being in this class, they need not necessarily be so. For example, if two opponents are both seeking to close with each other, they may both co-operate to minimize the time taken to do so. Thus a more realistic model should be partly co-operative (i.e. non zero sum). Games of that type are considerably more complicated, and thus it is felt that results based on a zero sum model are a necessary first step. The information assumption is unrealistic under any circumstances; however, it is felt that obtaining a workable technique based on perfect information is a necessary simplification in the initial investigation of the payoff.

Dynamic Model

In modeling the aircraft dynamics, the common assumption of a point mass, a flat earth, and constant gravity are made. In addition, the speed and altitude of each aircraft is also held constant. The last two are very limiting, but work has indicated that including these as variables in the formulation increases the number of dimensions to a point that makes interpretation of results vary difficult. (Refs 6:47, 9:115). These references also indicate that for a given game formulation, simple dynamic models give a fair indication of what to expect

from more complicated dynamics (Refs 6:46, 9:193).

State and Control Variables

The above assumptions result in a six dimensional unbounded game space consisting, for each aircraft, of two position co-ordinates in a constant altitude horizontal plane, and a heading. Six dimensions lead to relatively straightforward dynamic equations; however, the greater the number of variables, the more difficult it becomes to interpret the results. What is required for interpretation is as few state variables as possible, having close correspondence to the states which pilots actually use. With a relative co-ordinate system on one aircraft, the number of states is reduced to three.

The aircraft are designated A and B, and a body fixed rotating co-ordinate system is located on A, with its X axis in the direction of the longitudinal axis through A's nose. In this dynamic model this direction corresponds to the direction of A's velocity at a given time. As shown in Fig. 1, three independent state variables remain. The state vectors used are $\{x(t), y(t), z(t)\}$ and $\{r(t), \theta(t), z(t)\}$, and it is convenient to use both in the development.

The controls are the rates of turn of A and B, $U_A(t)$ and $U_B(t)$, respectively. Each is assumed to be constrained in either direction by a constant maximum turn rate, U_{AMAX} and U_{BMAX}, respectively. To be realistic, a model should include instantaneous and sustained max-rate capability,

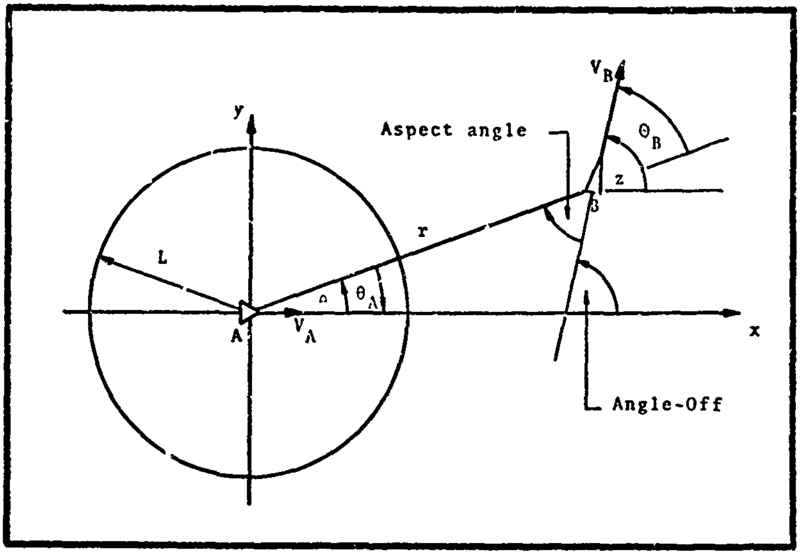

Fig. 1. Co-ordinate System

as well as changes due to limits on weapon tracking and release (Ref 12:3). However, in relation to the simplified aircraft dynamics, constant maximum turn rates are reasonable.

The state equations of motion are thus

$$\dot{x} = V_B \cos z - V_A + U_A y \tag{2-1}$$

$$\dot{y} = V_B \sin z - U_A x \tag{2-2}$$

$$\dot{z} = U_B - U_A \tag{2-3}$$

and

$$\dot{r} = V_B \cos (z-\theta) - V_A \cos \theta \tag{2-4}$$

$$\dot{\theta} = \frac{1}{r} (V_B \sin (z-\theta) + V_A \sin \theta) - U_A \tag{2-5}$$

$$\dot{z} = U_B - U_A \tag{2-6}$$

where $V_A \triangleq$ speed of A

$V_B \triangleq$ speed of B

and both V_A and V_B are constant.

The Terminal Constraint

The conditions needed for a successful conclusion by either A or B (and conversely failure for the other) depend largely on their respective weapon systems. The most common condition used to end games is a fixed final range from the target. In the planar case, this generates a circle of radius L (the weapon range) in the xy plane around one of the combatants (Fig. 1, page 8). Other approaches have been used to create more realistic terminal conditions: more complex areas such as elliptical kill probability regions rotating behind each combatant (Ref 6:7), and simple ones such as "gun spikes" pointing directly forward of each aircraft (Ref 10). Use of elliptical regions in the role problem requires two such areas rotating behind the respective aircraft; this rapidly becomes unworkable. The "gun spike" concept has been used for role determination with a range payoff, but suffers from being a "boresight only" condition.

The popularity of a circular terminal condition is partly due to the relative ease of obtaining closed form solutions, and it is largely for that reason it is used in this study. The condition is defined as

$$x_f^2 - y_f^2 - L^2 = \psi = 0 \qquad (2\text{-}7)$$

$$r_f^2 - L^2 = \psi = 0 \tag{2-8}$$

where (x_f, y_f, z_f) and (r_f, θ_f, z_f) are the final states and L is the weapon range. This constraint has to be satisfied for the game to end; how it ends, however, is determined by the payoff.

The Payoff

For this class of game, the payoff (or cost function) represents the objective of the players throughout the game. It is thus fundamental to question of role. Lynch's general purpose payoff, with A minimizing and B maximizing is

$$J = a \cos^2\left(\frac{\theta_B}{2}\right)_{tf} + b \sin^2\left(\frac{\theta_A}{2}\right)_{tf} + c \cos \theta_{B_{tf}} \int dt \tag{2-9}$$

where a, b, c are constant weighting factors and θ_A, θ_B are the angles between the velocity vectors of A and B and the position vector between them (Fig. 1, page 8).

Changing the form of Eq (2-9) and rewriting in terms of the polar co-ordinate system (r, θ, z), the payoff becomes

$$J = \frac{(a+b)}{2} - \frac{1}{2}\left[a \cos (z_f - \theta_f) + b \cos \theta_f\right] + c \cos (z_f - \theta_f) \int_{to}^{tf} dt \tag{2-10}$$

The middle terms represents the attempt by each player to end the game (be it success or failure) with his best possible combination of angle off and aspect angle. The last term indicates that when $|z_f - \theta_f|$ is small, then time becomes a critical

factor. Increasing the importance of this term is intended to polarize the roles of each player when termination is close, by the pursuer seeking to terminate quickly, and the evader trying to delay. The payoff was investigated in Reference 9 for two simple dynamic models, with a=b=0. The emphasis in this work is the case were c=0 (time is not included explicitly in the payoff).

<u>Outcomes to the Game</u>

There are four possible outcomes to a combat (Ref 3:61); letting $\underline{x}$ denote the state of the game, these outcomes are described by the terminal constraints

$$\underline{x} \in \psi_A \rightarrow \text{A is successful and destroys B} \quad (2\text{-}11)$$

$$\underline{x} \in \psi_B \rightarrow \text{B is successful and destroys A} \quad (2\text{-}12)$$

$$\underline{x} \in \psi_A \cap \psi_B = \psi_{AB} \rightarrow \text{Both A and B are destroyed (mutual kill)} \quad (2\text{-}13)$$

$$\underline{x} \in \overline{\psi_A \cup \psi_B} = \psi_D \rightarrow \text{Neither A nor B destroyed (draw)} \quad (2\text{-}14)$$

where ψ_A and ψ_B are the sets of states corresponding to successful outcomes for A and B respectively. In its basic form, the game terminates the first time $\underline{x}$ enters either ψ_A, ψ_B, or ψ_{AB}. If it enters none of these sets (i.e. $\underline{x} \in \psi_D$), there is a draw (the result of the game is inconclusive).

For player A, outcome (2-11) is obviously most preferable, and outcome (2-12) is least preferable. The preference ordering of outcomes (2-13) and (2-14) is not so obvious (Ref 3:64).

Using the single general purpose payoff J to represent the outcome criteria in a zero sum game, the outcomes (A

minimizing and B maximizing) are defined as:

$$J \le J_A \rightarrow \underline{x}_f \in \psi_A \rightarrow \text{success for A} \qquad (2\text{-}15)$$

$$J \ge J_B \rightarrow \underline{x}_f \in \psi_B \rightarrow \text{success for B} \qquad (2\text{-}16)$$

$$J_A < J < J_B \rightarrow \underline{x}_f \in \overline{\psi_A \cup \psi_B} \rightarrow \text{draw} \qquad (2\text{-}17)$$

where $J_{min} \le J \le J_{max}$ and $J_B \ge J_A$
the mutual kill outcome will not occur with this payoff unless $J_A = J_B$.

Eqs (2-15) to (2-17) enable a game of kind to be formulated. The problem of solution in the large is to determine which of the outcomes result from any set of starting points in the game space.

III. Problem Formulation

The purpose of this chapter is to formulate a differential game on the basis of Chapter II and the theory summarized in Appendix A. The Lynch payoff is used to establish a game of kind, and the influence of the weighting factors in the payoff is considered. The necessary conditions are applied, and possible control combinations are derived.

Payoff Formulation

The Lynch payoff of Eq (2-10) can be expressed as

$$J = P + Q \int_{to}^{tf} dt \qquad (3\text{-}1)$$

where $P \triangleq \frac{a+b}{2} - \frac{1}{2} (a \cos (z_f - \theta_f) + b \cos \theta_f)$

and $Q = c \cos (z_f - \theta_f)$

As time is not usually the overriding factor in combat (apart from total engagement limitations), it is not considered here explicitly in the payoff. With c=0, the payoff becomes

$$J = \phi = \frac{a+b}{2} - \frac{1}{2} (a \cos (z_f - \theta_f) + b \cos \theta_f) \qquad (3\text{-}2)$$

where z_f is the angle off, and $(z_f - \theta_f)$ is the aspect angle of B w.r.t. A (Fig. 1, page 8).

The relationship of the payoff to the terminal surface of Eq (2-8) is shown in Fig. 2 (page 14). Linesof constant payoff (isocosts) are drawn on a flattened terminal surface for a=b=1. The isocosts enclose two areas, one around A's best success point, and the other around B's. If limits are set on the payoff, a game of kind is established, and

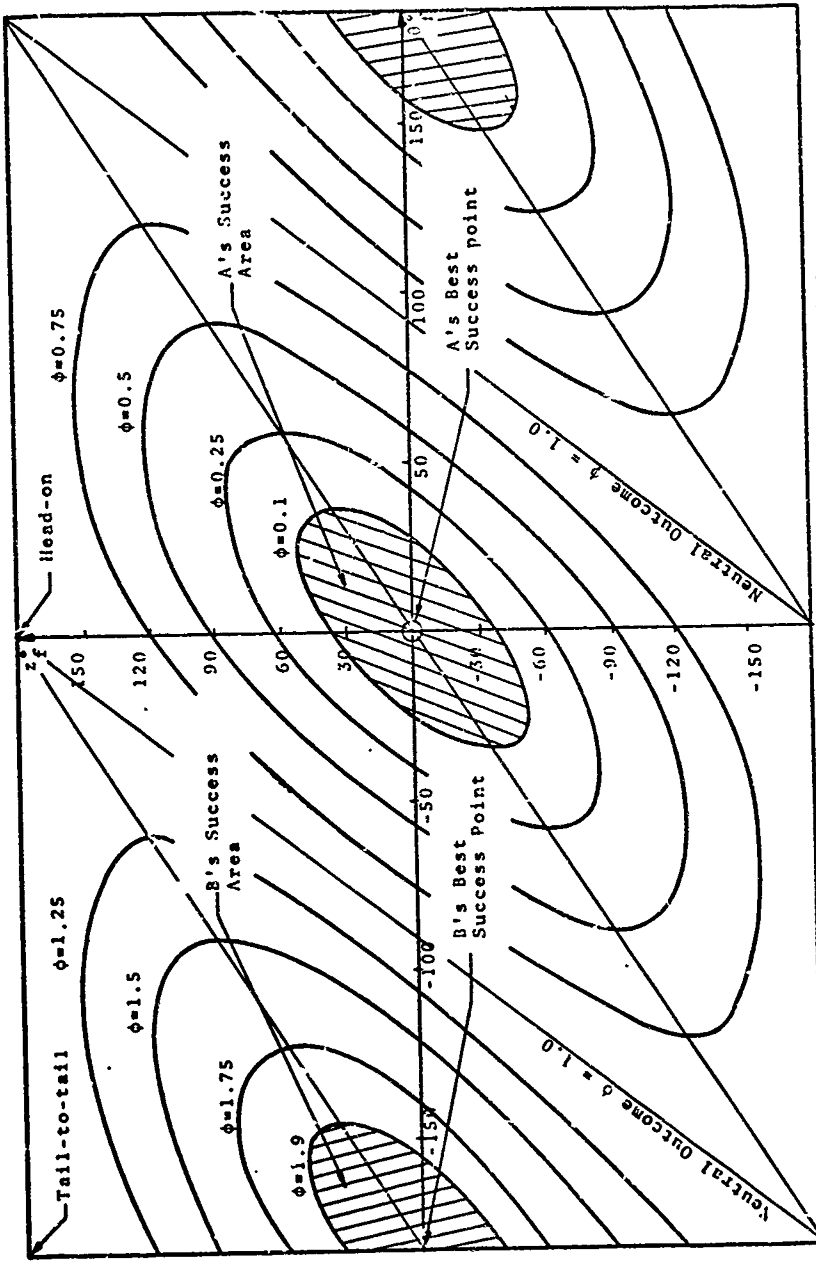

Fig. 2. Isocosts on the Terminal Surface for a=b=1

the isocosts for those limits enclose areas that represent the allowable z_f and θ_f combinations for a successful outcome by A an B. Using Eqs (2-15) to (2-17), in terms of ϕ, the outcomes (A minimizing and B maximizing) are

$$0 \leq \phi \leq \phi_A \rightarrow \text{success for A} \tag{3-3}$$

$$\phi_B \leq \phi \leq a+b \rightarrow \text{success for B} \tag{3-4}$$

$$\phi_A < \phi < \phi_B \rightarrow \text{inconclusive draw} \tag{3-5}$$

where ϕ_A and ϕ_B may be set to reflect the upper limits of A and B's weapon system at terminal range. The neutral outcome ϕ_N is given by

$$\phi_N = \frac{a+b}{2} \tag{3-6}$$

and includes head-on and tail-to-tail outcomes as draws (Fig. 2, page 14).

The size of the success areas can be adjusted by the limits set on the payoff (ϕ_A and ϕ_B), and their shape can be adjusted by the weighting factors a and b. Increasing a, increases the relative importance of $(z_f - \theta_f)$: and similarily, increasing b, increases the relative importance of θ_f. The effect this has on the shape of success areas is shown in Fig. 3 (page 16). In this way, the payoff can be adjusted to represent the angular conditions required at termination for a wide range of weapon systems. For example, consider that both A and B require a close approximation to boresight for their particular weapon systems, and have to be within a certain aspect angle to the rear of their opponent. With the

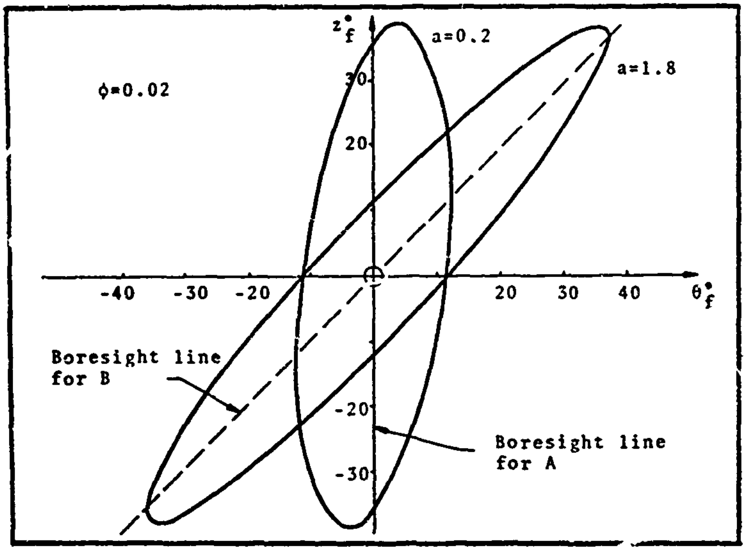

Fig. 3. Variation of Isocosts with Weighting Factors

co-ordinate system used, this implies A would require only a small variation in $|\theta_f|$, implying a large value of b in the payoff; whereas B would require only a small variation in $|z_f - \theta_f|$, implying a large value of a in the payoff. This is shown in Fig. 4. It is interesting that because of the use of a rotating co-ordinate system, the form of the payoff for success by A and B is different, even though their conditions for success are the same. Within the limits of this study, this difference, and the implied lack of a clearly defined neutral outcome, is not considered explicitly.

Application of Necessary Conditions

The Hamiltonian is formed from Eqs (2-1) to (2-6) as

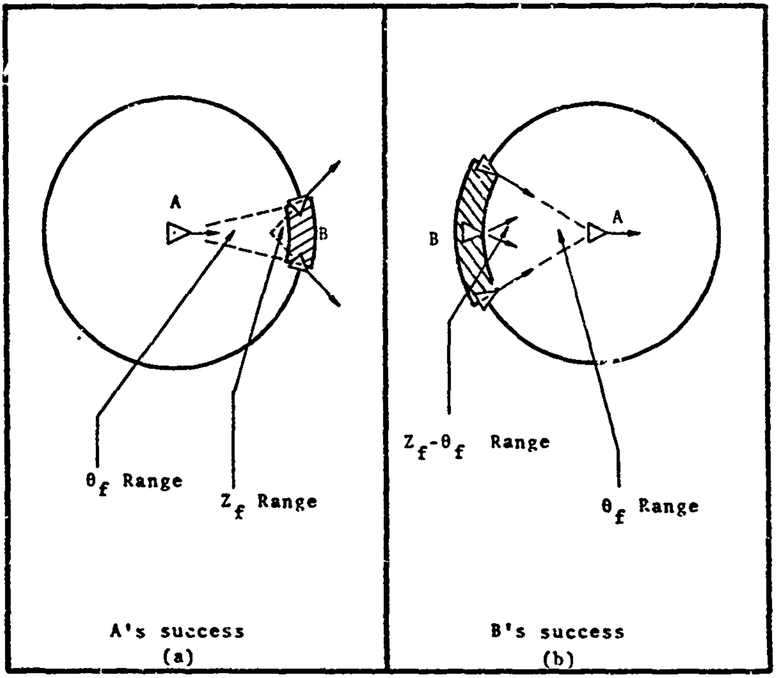

Fig. 4. Close to Boresight Termination for A and B

$$H = H_A + H_B + H_0 \tag{3-7}$$

where $H_A = S_A U_A$ and $S_A = \lambda_x y - \lambda_y x - \lambda_z = -\lambda_\theta - \lambda_z$ (3-8)

$$H_B = S_B U_B \text{ and } S_B = \lambda_z \tag{3-9}$$

$$\begin{aligned} H_0 &= V_B\,(\lambda_x \cos z + \lambda_y \sin z) - \lambda_x V_A \\ &= V_B\,(\lambda_r \cos (z - \theta) + \frac{\lambda_\theta}{r} \sin (z - \theta)) + \\ &\quad V_A(\frac{\lambda_\theta}{r} \sin \theta - \lambda r \cos \theta) \end{aligned} \tag{3-10}$$

<u>Co-states</u>. The co-state equations are

$$\dot{\lambda}_x = U_A \lambda_y \tag{3-11}$$

$$\dot{\lambda}_y = -U_A \lambda_x \tag{3-12}$$

$$\dot{\lambda}_z = V_B(\lambda_x \sin z - \lambda_y \cos z) \tag{3-13}$$

and in polar form

$$\dot{\lambda}_r = \frac{\lambda_\theta}{r}(V_A \sin \theta + V_B \sin (z - \theta)) \tag{3-14}$$

$$\begin{aligned}\dot{\lambda}_\theta = & -\lambda_r(V_A \sin \theta + V_B \sin (z - \theta)) - \\ & \frac{\lambda_\theta}{r}(V_A \cos \theta - V_B \cos (z - \theta))\end{aligned} \tag{3-15}$$

$$\dot{\lambda}_z = \lambda_r V_B \sin (z - \theta) - \frac{\lambda_\theta}{r} V_B \cos (z - \theta) \tag{3-16}$$

<u>Transversality</u>. Adjoining the constraint of Eqs (2-7) and (2-8) to the payoff of Eq (3-2) gives the transversality conditions.

$$\lambda_{x_f} = -\frac{y_f^2}{2L^3}\left((a \cos z_f + b) - \frac{x_f}{y_f} a \sin z_f\right) + \nu x_f \tag{3-17}$$

$$\lambda_{y_f} = \frac{x_f^2}{2L^3}\left(\frac{y_f}{x_f}(a \cos z_f + b) - a \sin z_f\right) + \nu y_f \tag{3-18}$$

$$\lambda_{z_f} = \frac{a}{2L}(x_f \sin z_f - y_f \cos z_f) \tag{3-19}$$

and in polar form

$$\lambda_{r_f} = \nu L \tag{3-20}$$

$$\lambda_{\theta_f} = \frac{1}{2} b (\sin \theta_f - a \sin (z_f - \theta_f)) \tag{3-21}$$

$$\lambda_{z_f} = \frac{1}{2} a \sin (z_f - \theta_f) \tag{3-22}$$

Also, as time is free

$$H(tf) = H(t) = 0 \qquad (3\text{-}23)$$

The unknown multiplier ν can be eliminated (Appendix B page 77) to give

$$\lambda_{y_f} = [L \sin \zeta_f (U_A b \sin \theta_f - U_B a \sin (z_f - \theta_f)) + (V_B \cos z_f - V_A)(b \sin \theta_f - a \sin (z_f - \theta_f))] / 2L(V_B \cos (z_f - \theta_f) - V_A \cos \theta_f) \qquad (3\text{-}24)$$

$$\lambda_{x_f} = [2L \lambda_{y_f} \cos \theta_f - (b \sin \theta_f - a \sin (z_f - \theta_f))] / 2L \sin \theta_f \qquad (3\text{-}25)$$

Non-Singular Controls. H is linear in U_A and U_B so singular arcs are possible: for the non-singular case $\min_A \max_B$ H gives the possible controls σ_A, σ_B as

$$\sigma_A = -(\text{sgn } S_A)\, U_{A\max} \qquad (3\text{-}26)$$

$$\sigma_B = (\text{sgn } S_B)\, U_{B\max} \qquad (3\text{-}27)$$

Singular Controls. Necessary conditions for A to have a singular arc give

$$S_A = 0 \rightarrow \lambda_x y = \lambda_z \text{ and } \lambda_z = -\lambda_\theta \qquad (3\text{-}28)$$

$$\dot{S}_A = 0 \rightarrow \lambda_y = 0 \text{ and } \lambda_\theta = -r\lambda_r \tan \theta \qquad (3\text{-}29)$$

$$\ddot{S}_A = 0 \rightarrow \sigma_A = 0 \qquad (3\text{-}30)$$

$$(S_A^{\cdot\cdot})_{\sigma_A} = -V_A \lambda_x + \lambda_x \geq 0 \qquad (3\text{-}31)$$

hence A may have a singular control of zero turn rate. Similarly, the application of singular necessary conditions for B give

$$S_B = 0 \rightarrow \lambda_z = 0 \qquad (3\text{-}32)$$

$$\dot{S}_B = 0 \rightarrow \lambda_y = \lambda_x \tan z \text{ and } \lambda_\theta = r\,\lambda_r \tan(z-\theta) \qquad (3\text{-}33)$$

$$\ddot{S}_B = 0 \rightarrow \sigma_B = 0 \qquad (3\text{-}34)$$

$$(S_B^{\cdot\cdot})_{\sigma_B} = \frac{V_B \lambda_x}{\cos z} \quad \lambda_x \geq 0 \text{ for } \frac{-\pi}{2} < z_f < \frac{\pi}{2} \qquad (3\text{-}35)$$

$$\lambda_x \leq 0 \text{ for } \frac{\pi}{2} < z_f < \frac{-\pi}{2}$$

Hence B may also have a singular control of zero turn rate. If the conditions of Eqs (3-28) to (3-35) are all satisfied, then both A and B may be singular at the same time. Satisfying these conditions requires that y(t) = 0 and z(t) = 0, 180°. Consequently, a direct tail chase, a head-on or a tail-to-tail are the only situations where both may be singular.

<u>Control Combinations</u>. The non-singular controls represent maximum rates of turn for A and B in either direction, the singular controls represent straight dashes. There are thus a total of nine possible control combinations ($\sigma_A = \pm U_{Amax}$, 0 with $\sigma_B = \pm U_{Bmax}$, 0).

IV. Closed Form Solutions

Analysis is very much easier if as much of the game as possible is solved analytically. This chapter presents closed form results for the states and co-states along continuous control arcs, in terms of the time-to-go, and the final states and co-states. To solve the game fully it is necessary to identify the various surfaces separating these arcs. Some closed form expressions are derived for these surfaces. Finally these solutions are related to the terminal surface and payoff. Details of the derivations are contained in Appendix B.

States and Co-states

Over arcs on which the controls σ_A and σ_B are constant, the state and co-state equations are linear and can be solved fairly easily, at least in the cartesian form (Appendix B pages 77 to 81), to give, for non-singular arcs ($\sigma_A = \pm U_{Amax}$, $\sigma_B = \pm U_{Bmax}$)

$$x(t) = x_f \cos \sigma_A t + y_f \sin \sigma_A t + R_B [\sin (\sigma_A t - z_f) + \sin ((\sigma_B - \sigma_A) t + z_f)] - R_A \sin \sigma_A t \quad (4\text{-}1)$$

$$y(t) = -x_f \sin \sigma_A t + y_f \cos \sigma_A t + R_B [\cos (\sigma_A t - z_f) - \cos ((\sigma_B - \sigma_A) t + z_f)] + R_A (1 - \cos \sigma_A t) \quad (4\text{-}2)$$

$$z(t) = (\sigma_B - \sigma_A)\, t + z_f \tag{4-3}$$

$$\lambda_x(t) = \lambda_{x_f} \cos \sigma_A t + \lambda_{y_f} \sin \sigma_A t \tag{4-4}$$

$$\lambda_y(t) = -\lambda_{x_f} \sin \sigma_A t + \lambda_{y_f} \cos \sigma_A t \tag{4-5}$$

$$\lambda_z(t) = \lambda_{z_f} - 2R_B \sin \frac{\sigma_B t}{2} \cdot [\lambda_{y_f} \cos (\frac{\sigma_B t}{2} + z_f) + \lambda_{x_f} \sin (\frac{\sigma_B t}{2} + z_f)] \tag{4-6}$$

where $R_A = V_A/\sigma_A$ and $R_B = V_B/\sigma_B$, the radii of turn of A and B respectively, and the final time at the end of a given arc is zero, with time backwards along the arc being negative ($t_o < 0$, $t < 0$, $t_f = 0$).

For singular arcs modified solutions are generated; for singular A ($\sigma_A = 0$, $\sigma_B = \pm U_{Bmax}$)

$$x(t) = x_f - V_A t + R_B (\sin (z_f + \sigma_B t) - \sin z_f) \tag{4-7}$$

$$y(t) = y_f - R_B (\cos (\sigma_B t + z_f) - \cos z_f) \tag{4-8}$$

$$z(t) = z_f + \sigma_B t \tag{4-9}$$

$$\lambda_x(t) = \lambda_{x_f} \tag{4-10}$$

$$\lambda_y(t) = \lambda_{y_f} = 0 \tag{4-11}$$

$$\lambda_z(t) = \lambda_{z_f} - 2R_B \lambda_{x_f} \sin (\frac{\sigma_B t}{2} + z_f) \sin \frac{\sigma_B t}{2} \tag{4-12}$$

and for singular B ($\sigma_A = \pm U_{Amax}$, $\sigma_B = 0$)

$$x(t) = x_f \cos \sigma_A t + (y_f - R_A) \sin \sigma_A t + V_B t \cos (z_f - \sigma_A t) \tag{4-13}$$

$$y(t) = -x_f \sin \sigma_A t + (y_f - R_A) \cos \sigma_A t + V_B t \sin (z_f - \sigma_A t) + R_A \tag{4-14}$$

$$z(t) = z_f - \sigma_A t \tag{4-15}$$

$$\lambda_x(t) = \lambda_{x_f} \cos \sigma_A t + \lambda_{y_f} \sin \sigma_A t \tag{4-16}$$

$$\lambda_y(t) = -\lambda_{x_f} \sin \sigma_A t + \lambda_{y_f} \cos \sigma_A t \tag{4-17}$$

$$\lambda_z(t) = \lambda_{z_f} = 0 \tag{4-18}$$

Using Eqs (4-1) to (4-18) nine different arcs can be constructed from the nine possible combinations of control.

Game Surfaces

The game space is partitioned by surfaces which represent discontinuities in the controls and junctions between the various types of arcs. For a full solution to the game it is necessary to identify these surfaces, and the regions they partition. Appendix A discusses some of the more important surfaces, and the conditions necessary for their solution.

Switching Surfaces. Switching of A's controls (other than to zero) occurs when $S_A = 0$ and $\dot{S}_A \neq 0$. Similarly switching for B (other than to zero) occurs when $S_B = 0$ and $\dot{S}_B \neq 0$. Each of these conditions defines a five dimension surface in the $(x-\lambda)$ space, which is reduced to four dimensions by the free time conditions of Eq (3-23). For a given arc, a closed form solution can be found that gives the switching time (backward from the final conditions) in terms of the final states and co-states. The switching time is then a parameter that specifies a point in the switching surface.

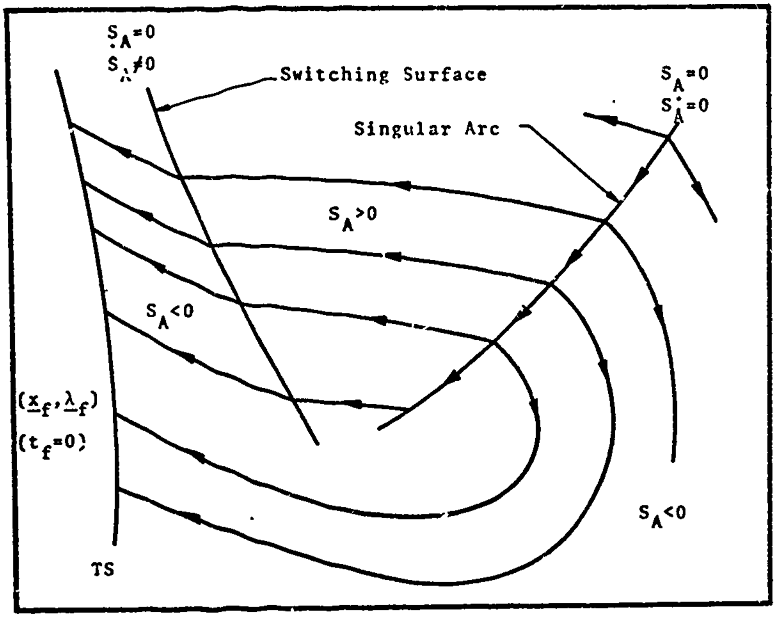

Fig. 5. Switching Surfaces and Singular Arcs

Fig. 5. shows a two dimensional illustration of a switching surface for A.

A's Switching Suface. For an A switching surface, the condition $S_A = 0$ gives

$$\lambda_{x_s} y_s - \lambda_{y_s} x_s - \lambda_{z_s} = 0 \tag{4-19}$$

where the states and co-states of a junction point between an arc and the surface are $(\underline{x}_s, \underline{\lambda}_s)$. Substitution in Eq (4-19) of the closed form state and co-state solutions for an arc gives

$$\begin{aligned} \lambda_{x_f}[y_f - R_A(1 - \cos\sigma_A t_s)] - \lambda_{y_f}[x_f - \\ R_A \sin\sigma_A t_s] - \lambda_{z_f} = 0 \end{aligned} \tag{4-20}$$

where t_s is the time (negative) at which the control switches going backwards along the arc ($t_s < 0$, $t_f = 0$). Hence Eq (4-20) and Eqs (4-1) to (4-3) are a parametric solution for the switching surface. The additional free time requirement of $H = 0$ in Eq (3-23) gives

$$(\lambda_{x_s} y_s - \lambda_{y_s} x_s)\, \sigma_B + V_B\, (\lambda_{x_s} \cos x_s + \lambda_{y_s} \sin x_s) - \lambda_{x_s} V_A = 0 \tag{4-21}$$

which, on substitution of Eqs (4-1) to (4-6) reduces to

$$R_B [\lambda_{y_f} (v \sin \sigma_A t_s - \sin z_f) + \lambda_{x_f} (v \cos \sigma_A t_s - \cos z_f)] - \lambda_{z_f} = 0 \tag{4-22}$$

where $v = V_A/V_B$, the speed ratio of A to B.

<u>B's Switching Suface</u>. For a B switching surface, the condition $S_B = 0$ gives

$$\lambda_{z_s} = 0 \tag{4-23}$$

Using the solution for $\lambda_z(t)$ of Eq (4-6), in terms of the final states and co-states and the switching time, the B switching surface is given by

$$\lambda_{z_f} + R_B [\lambda_{y_f} (\sin z_f - \sin (\sigma_B t_s + z_f)) + \lambda_{x_f} (\cos z_f - \cos (\sigma_B t_s + z_f))] = 0 \tag{4-24}$$

The additional free time condition of Eq (3-23) must also hold for the B switching surface and gives

$$(\lambda_{x_s} y_s - \lambda_{y_s} x_s)\, \sigma_A + (V_B (\lambda_{x_s} \cos x_s) + \lambda_{y_s} \sin x_s) - \lambda_{x_s} V_A = 0 \tag{4-25}$$

or in terms of the final conditions and t_s

$$-x_f\lambda_{y_f} + y_f\lambda_{x_f} + R_B[Y + X] - R_A\lambda_{x_f} = 0 \qquad (4\text{-}26)$$

where $X = \lambda_{x_f}\ (\cos z_f + (\sigma_r - 1)\cos(\sigma_B t_s + z_f))$

$Y = \lambda_{y_f}\ (\sin z_f + (\sigma_r - 1)\sin(\sigma_B t_s + z_f))$

or $\sigma_r = \sigma_B/\sigma_A$, the turn ratio of B to A

<u>Singular Surfaces</u>. Singular surfaces are reduced forms of switching surfaces; in addition to the switching functions being zero, at a junction with a singular surface it is also necessary that the conditions of Eqs (3-29) to (3-31) are satisfied for A (Eqs (3-33) to (3-35) for B). Arcs that satisfy these conditions and switch to singular controls at a point on the singular surface will remain on it (going backwards in time), until deliberately forced off (Fig. 5, page 24). Conversely, going forward, an infinite number of trajectories join the singular arc. The additional condition of $H = 0$ reduces the dimension of the surface to three, and thus a surface in the state space.

<u>A's Singular Surface</u>. For an A singular surface, Eq (3-29) gives

$$\lambda_{y_s} = 0 \qquad (4\text{-}27)$$

Using the solution for $\lambda_y(t)$ given in Eq (4-5), in terms of the final conditions and the switching time the singular condition becomes

$$\lambda_{x_f}\sin\sigma_A t_s - \lambda_{y_f}\cos\sigma_A t_s = 0 \qquad (4\text{-}28)$$

The free time requirement still exists; substituting

Eqs (3-28) to (3-30) into H = 0 gives

$$\frac{y_s}{R_B} + \cos z_s - v = 0 \qquad (4\text{-}29)$$

This is the equation in state space of the A singular surface. Arcs that intersect this surface and satisfy the singular switching conditions of Eq (3-28) to Eq (3-31) switch to singular A control, and remain on the surface, with their trajectories given by Eqs (4-7) to (4-12). In terms of the final conditions and switching time, Eq (4-29) becomes

$$-(V_A/\sigma_B) - x_f \sin \sigma_A t_s + y_f \cos \sigma_A t_s +$$

$$R_B \cos (\sigma_A t_s - z_f) + R_A \sin (1 - \cos \sigma_A t_s) = 0 \qquad (4\text{-}30)$$

<u>B's Singular Surface</u>. For a B singular surface, Eqs (3-32) and (3-33) give

$$\lambda_{z_s} = 0 \qquad (4\text{-}31)$$

and $$\lambda_{x_s} \sin z_s - \lambda_{y_s} \cos z_s = 0 \qquad (4\text{-}32)$$

which in terms of the final conditions and t_s become

$$\lambda_{z_f} = 0 \qquad (4\text{-}33)$$

and $$\lambda_{x_f} \sin (\sigma_B t_s + z_f) - \lambda_{y_f} \cos (\sigma_B t_s + z_f) = 0 \qquad (4\text{-}34)$$

As for A, the free time requirement must be satisfied, and substituting Eqs (3-32) to (3-34) into H = 0 gives a B singular surface in state space as

$$(y_s \cos z_s - x_s \sin z_s)R_A + (1/v) - \cos z_s = 0 \qquad (4\text{-}35)$$

In terms of the final conditions and t_s, the condition is

$$V_B/\sigma_A - [x_f \sin(\sigma_B t_s + z_f) - (y_f - R_A) \cos(\sigma_B t_s + z_f) + R_B (1 - \cos \sigma_B t_s)] = 0 \tag{4-36}$$

In Chapter III page 20 it was shown that the only possible conditions for both A and B to be singular at the same time were $v(t) = 0$ and $z(t) = 0$. Substitution of this into the equations for the singular surfaces (Eqs 4-25) and (4-35), with $\sigma_A = 0$ and $\sigma_B = 0$, gives $v = 1$. This implies that unless the speeds of A and B are the same, the joint singular case does not exist, and a direct tail chase, head-on, or tail-to-tail will not occur, other than instantaneously.

Dispersal Surfaces. From Appendix A, the major requirements of points on a dispersal surface are that two or more paths, with differing control combinations on each, intersect, and that the payoff at termination is the same for each path. For two paths this condition can be expressed as

$$\underline{x}_{s_1} = \underline{x}_{s_2} = \underline{x}_s \tag{4-37}$$

$$\phi_1 = \phi_2 = 0 \tag{4-38}$$

where $\underline{x}_{s_1}$, and $\underline{x}_{s_2}$ are the state vectors of each path at the dispersal surface, and ϕ_1 and ϕ_2 are their respective terminal payoffs. If paths travel directly from the dispersal surface to the terminal surface without switching controls, as shown in Fig. 6, then using the closed form solutions for states and co-states over each constant control arc, the dispersal surface can be expressed in terms of the

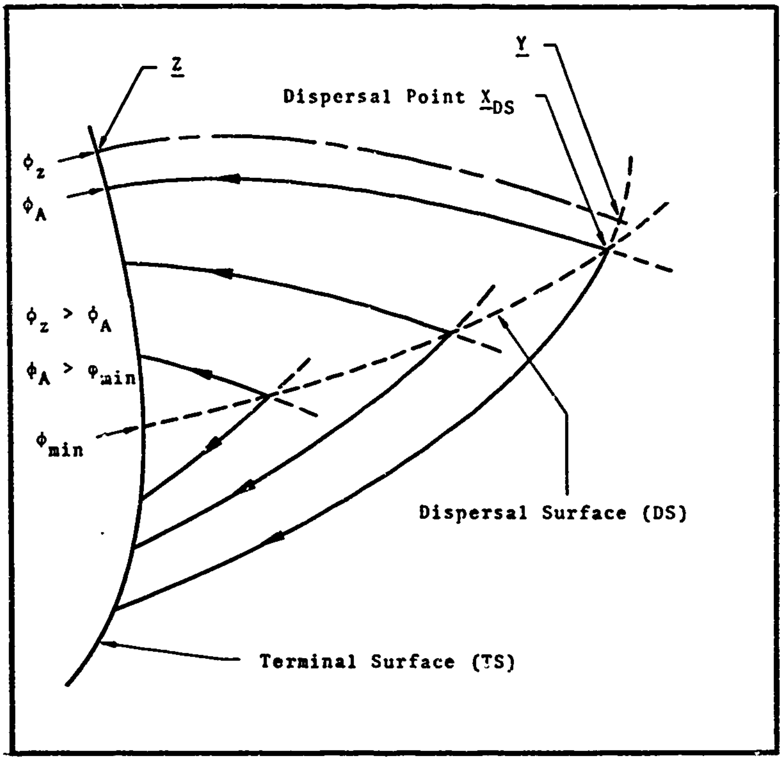

Fig. 6. Characteristics of Dispersal Surfaces

terminal conditions as

$$x_{2_s} - x_{1_s} = f_1(\underline{x}_{f_1}, \underline{x}_{f_2}, t_{1_s}, t_{2_s}) = 0 \qquad (4\text{-}39)$$

$$y_{2_s} - y_{1_s} = f_2(\underline{x}_{f_1}, \underline{x}_{f_2}, t_{1_s}, t_{2_s}) = 0 \qquad (4\text{-}40)$$

$$z_{2_s} - z_{1_s} = f_3(\underline{x}_{f_1}, \underline{x}_{f_2}, t_{1_s}, t_{2_s}) = 0 \qquad (4\text{-}41)$$

and

$$\phi_1 = \phi = f(\underline{x}_{f_1}) \qquad (4\text{-}42)$$

$$\phi_2 = \phi = f(\underline{x}_{f_2}) \qquad (4\text{-}43)$$

where t_{1_s} and t_{2_s} are the times of intersection along paths 1 and 2. Equations (4-39) to (4-43), together with the free time condition for each path, the terminal constraints, and the transversality conditions, fully define a dispersal surface. A more condensed form was not found; however, the above was used as the basis for a numerical search for the dispersal surface.

Solutions at the Terminal Surface

To complete the solution, trajectories and surfaces in the game space are analyzed backwards from the terminal surface; the first step is to find the controls at the TS, and any intersections with other surfaces.

Controls. For the controls at the TS, the condition of Eq (4-19) gives

$$S_A = \lambda_{x_f} y_f - \lambda_{y_f} x_f - \lambda_{z_f} = -\lambda_{\theta_f} - \lambda_{z_f} \qquad (4\text{-}44)$$

$$S_B = \lambda_{z_f} \qquad (4\text{-}45)$$

Substituting in the transversality conditions of Eqs (3-21) and (3-22) for the co-states, results in

$$S_A = -b/2 \sin \theta_f \qquad (4\text{-}46)$$

$$S_B = \frac{a}{2} \sin (z_f - \theta_f) \qquad (4\text{-}47)$$

Using the conditions of Eqs (3-26) and (3-27) the nonsingular controls on the TS are

$$\sigma_{A_f} = U_{AMAX} \text{ for } 0 < \theta_f < \pi$$

$$\sigma_{A_f} = U_{AMAX} \text{ for } -\pi < \theta_f < 0$$
$$\sigma_{A_f} = U_{BMAX} \text{ for } 0 < (z_f-\theta_f) < \pi \qquad (4\text{-}48)$$
$$\sigma_{B_f} = U_{BMAX} \text{ for } -\pi < (z_f-\theta_f) < 0$$

This control distribution is shown in Fig. 7 (page 32). As Lynch points out (Ref 9:191) the controls are the exact opposite of those produced in the classical barrier problem. In this case, with the generalized payoff, A is turning into line of sight A to B, and B is turning into line of sight B to A. Hence using this payoff, the terminal controls are aggressive. In a defensive position, each aircraft continues turning when at the opponents kill range, in an attempt to increase angle off and aspect angle and maneuver into an attacking position.

<u>TS Switching Surfaces</u>. The points on the TS where the switching functions S_A and S_B are zero are intersection points of the TS with switching surfaces. The lines of intersection thus formed are shown in Fig. 7 and are given by

$$S_A = 0 \rightarrow \text{A switching line of } \theta_f = 0, \pi; \text{ for } -\pi < z_f < \pi$$
$$S_B = 0 \rightarrow \text{B switching line of } (z_f-\theta_f) = 0, \pi; \text{ for } -\pi < \theta_f < \pi \qquad (4\text{-}49)$$

If $\dot{S}_A = 0$ or $\dot{S}_B = 0$, the possibility exists for singular arcs to terminate on these switching lines. For A, the singular condition of Eq (3-29) with $\theta_f = 0, \pi$ gives

$$\dot{S}_A = -V_A\lambda_{\theta_f}/L \qquad (4\text{-}50)$$

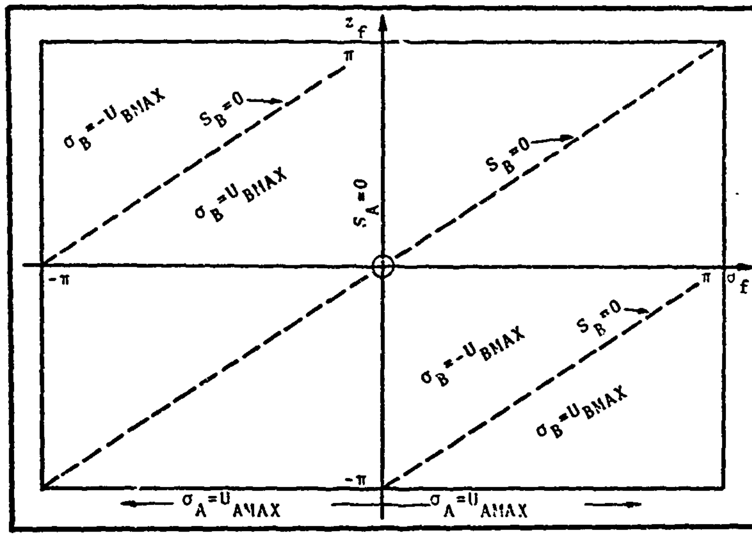

Fig. 7. Controls at the Terminal Surface

Substituting in the terminal conditions of Eq (3-21) results in

$$\dot{S}_A = \frac{aV_A \sin z_f}{2L} \qquad (4\text{-}51)$$

Using this, A's controls along the switching lines $\theta_f = 0, \pi$ are

$$\begin{aligned} \sigma_A(t_f^-) &= -U_{AMAX} \text{ for } 0 < z_f < \pi \\ \sigma_A(t_f^-) &= +U_{AMAX} \text{ for } -\pi < z_f < 0 \qquad . \qquad (4\text{-}52) \\ \sigma_A(t_f) &= 0 \text{ for } z_f = 0, \pi \end{aligned}$$

Hence the only co-ordinates at which an A singular arc can terminate at the TS are (0, 0), (0, π), (π, π) (π, 0). For B, the singular condition of Eq (3-33) with $(z_f - \theta_f) = 0, \pi$

gives

$$S_B^{\cdot} = \frac{-\lambda\theta_f V_B}{L} \tag{4-53}$$

Substituting in the terminal conditions of Eq (3-21) results in

$$S_B^{\cdot} = \frac{-bV_B \sin\theta_f}{2L} \tag{4-54}$$

and B's controls along the switching lines $z_f - \theta_f = 0, \pi$ are

$$\sigma_B(t_f^-) = +U_{BMAX} \text{ for } 0 < \theta_f < \pi$$

$$\sigma_B(t_f^-) = -U_{BMAX} \text{ for } -\pi < \theta_f < 0 \tag{4-55}$$

$$\sigma_B(t_f) = 0 \text{ for } \theta_f = 0, \pi$$

Hence the only co-ordinates on the TS at which a B singular arc could terminate are identical to those for an A singular arc. This implies that singular arcs only terminate at the TS when both A and B are singular. The further requirement of equal speed (page 28) virtually eliminates the possibility of singular arcs at the TS.

<u>TS Singular Surfaces</u>. The A and B singular surfaces of Eqs (4-29) and (4-35) may intersect with the TS for certain values of the game parameters. For A, substitution of the terminal constraint into Eq (4-29) gives

$$\frac{L \sin\theta_f}{R_B} + \cos z_f - v = 0 \tag{4-56}$$

for the intersection line. The position of the line will vary with L, R_B, and v as shown in Fig. 8 (page 34). This

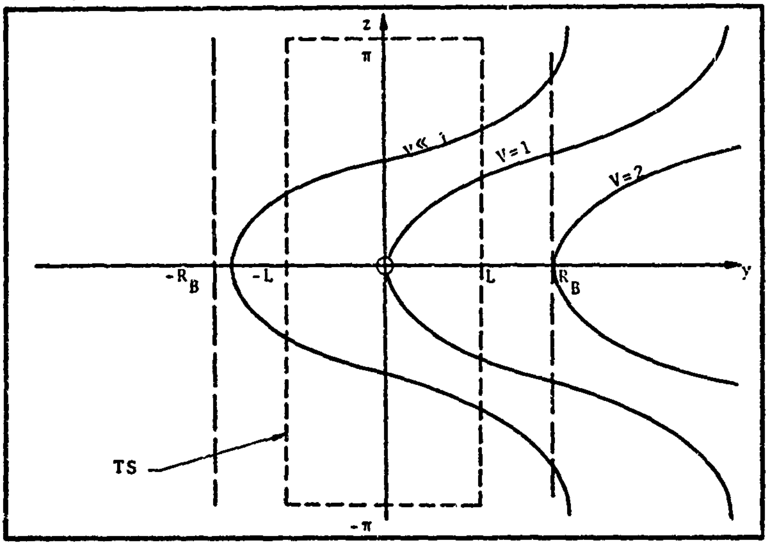

Fig. 8. Projection of A Singular Surfaces in zy plane

is because the singular surface given by Eqs (4-29) and (4-35) are necessary, but not sufficient, for singularity. In this case, a terminal cost gives transversality conditions which must also be satisfied for a singular arc. This reduces the A singular surface to a curve in the state space which is very unlikely to intersect the TS. A similar argument applies for the B singular surface.

<u>TS Dispersal Surfaces</u>. The switching lines for B given by Eq (4-48) also satisfy the conditions for a dispersal surface (in a limiting sense) given by Eqs (4-39) to (4-43). Thus the lines $z_f - \theta_f = 0, \pi$ are the intersection of a B dispersal surface, because only B has a choice of controls. Similarly, the only positions on the TS at which A has a dispersal condition are (0, 0), (0, π), (π, 0) and (π, π).

These co-ordinates also lie on the B dispersal line and thus represent points of "double choice" or "mixed strategy" between A and B.

V. Analysis of the Solution in the Large

To complete the full solution of the game, it is necessary to locate the game surfaces, and analyze the way in which they partition the game space. The method consists of constructing trajectories and surfaces backwards from the terminal surface. The application of this method using the closed form solutions is presented in this chapter.

Two techniques are used in the analysis:

1. Surface Solutions in Closed Form. The closed form solutions for surfaces, though mostly transcendental in the switching time, can be solved numerically. However, the solution does become very complex if the surfaces are connected to the terminal surface by trajectories consisting of a number of arcs.

2. Trajectory Analysis. Trajectories are constructed backwards from isocosts on the TS. The conditions of Chapter III pages 19 to 20 are used to switch controls, thus generating trajectories composed of a number of arcs. The global optimality of these trajectories is uncertain until surfaces can be isolated by parameterization and projection of a large number of trajectories (Refs 10, 11).

The advantage of the first technique is that it gives the surfaces directly, though with difficulty for surfaces not close to TS. The second method has the advantage that trajectories can be plotted to allow visual interpretation;

though this is very long and tedious, and of doubtful advantage for problems modeled with more than three dimensions.

General Trajectory Patterns

In order to select some typical parameters for more detailed analysis, the second technique was used to indicate the general pattern of trajectories for various game parameters. The following main observations result:

1. Singular arcs occur and do not intersect the TS, thus confirming the results of Chap IV page 33.

2. Some trajectories appear to contradict actual experience; for example, as shown in Fig. 9 (page 38). as B overshoots he is turning away from A, rather than inwards as might be expected.

3. As shown in Figs. 11 and 12 (pages 40 and 41), many successful terminations result from trajectories that correspond to a scissors maneuver (Ref 7:62).

4. To be as realistic as possible, and comparable with other work, specific parameters selected for more detailed analysis are

$$V_A = 800 \text{ ft/sec } U_{AMAX} = 0.2 \text{ rads/sec}$$

giving

$$R_A = 4000 \text{ feet}$$

and

$$V_B = 720 \text{ ft/sec } U_{BMAX} = 0.24 \text{ rads/sec}$$

giving

$$R_B = 3000 \text{ feet}$$

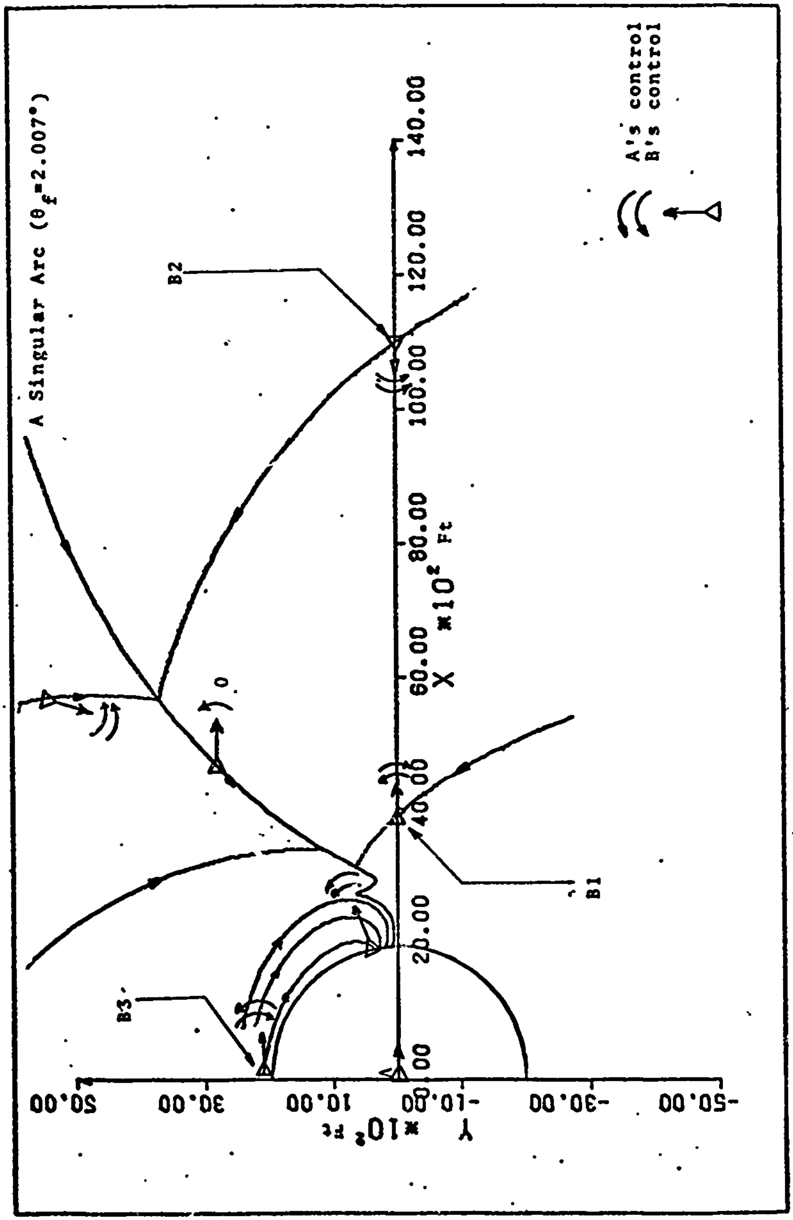

Fig. 9. Trajectories Ending in a Lag for A (θ_f> Singular A θ_f)

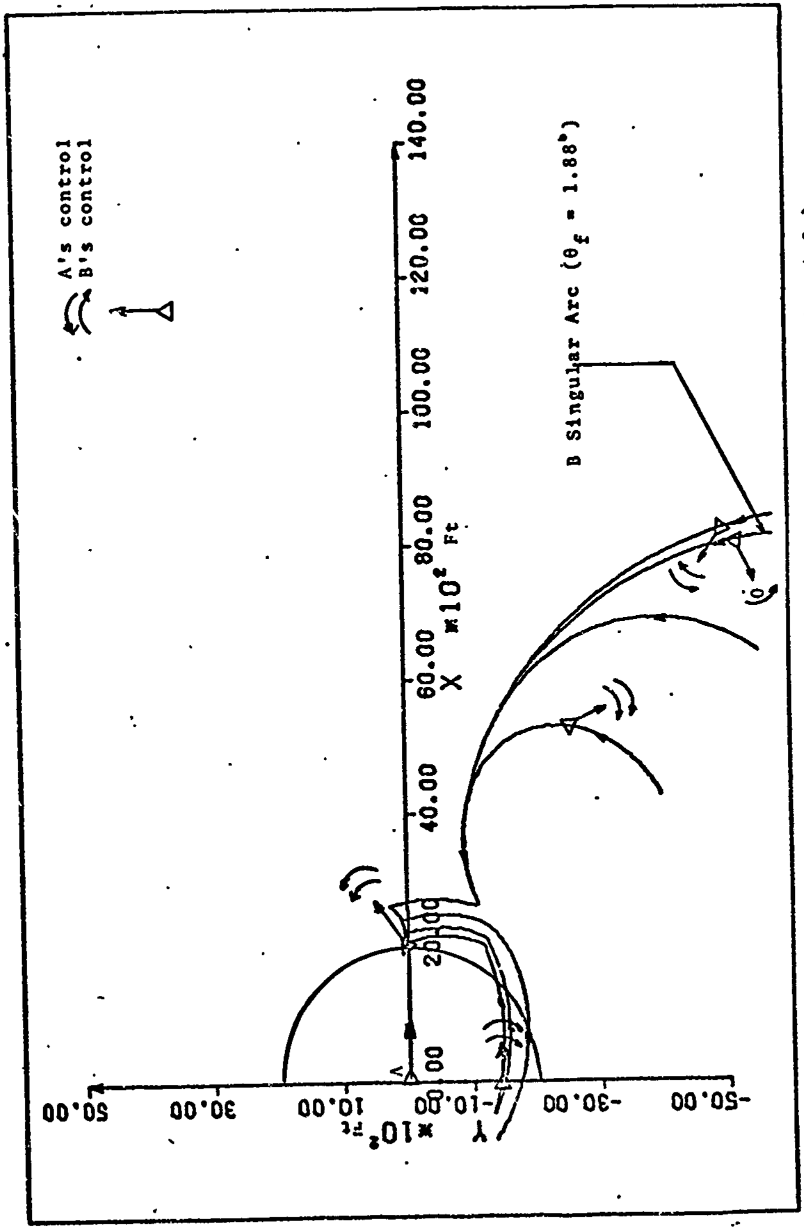

Fig. 10. Trajectories Ending in a Lag for A (θ_f < Singular A θ_f)

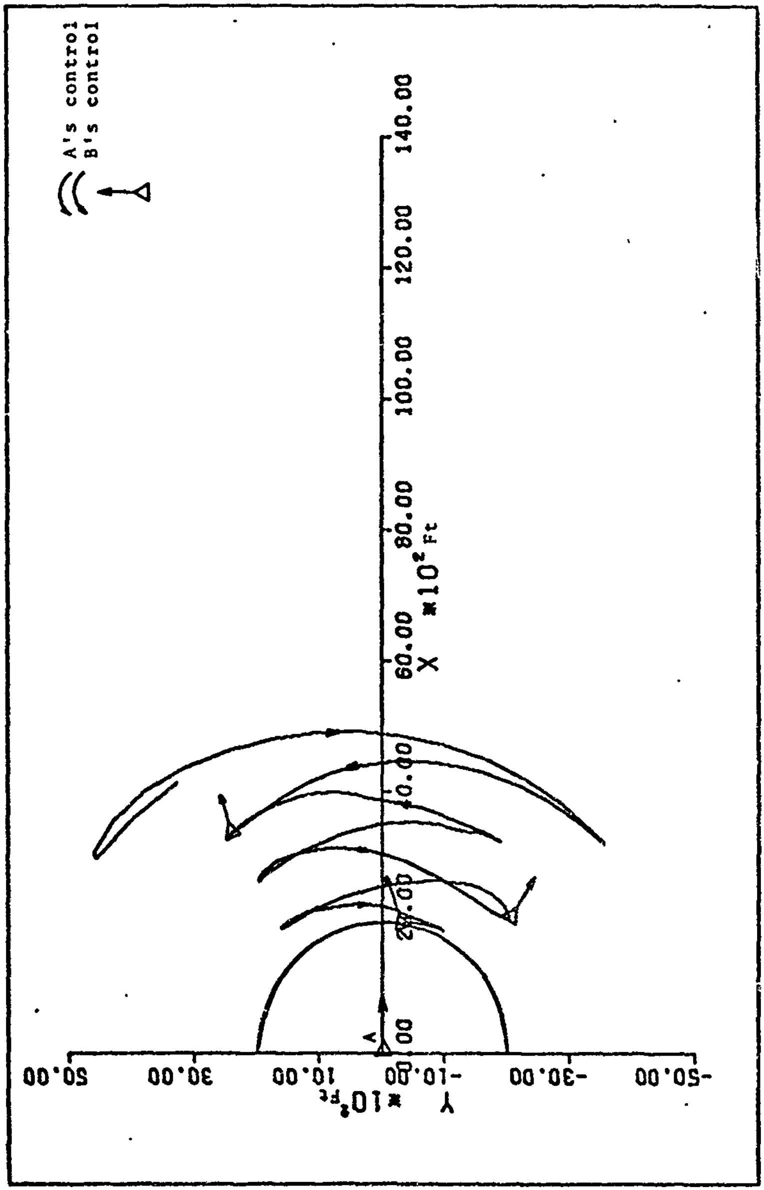

Fig. 11. Scissors Trajectory Ending in a Lead for A

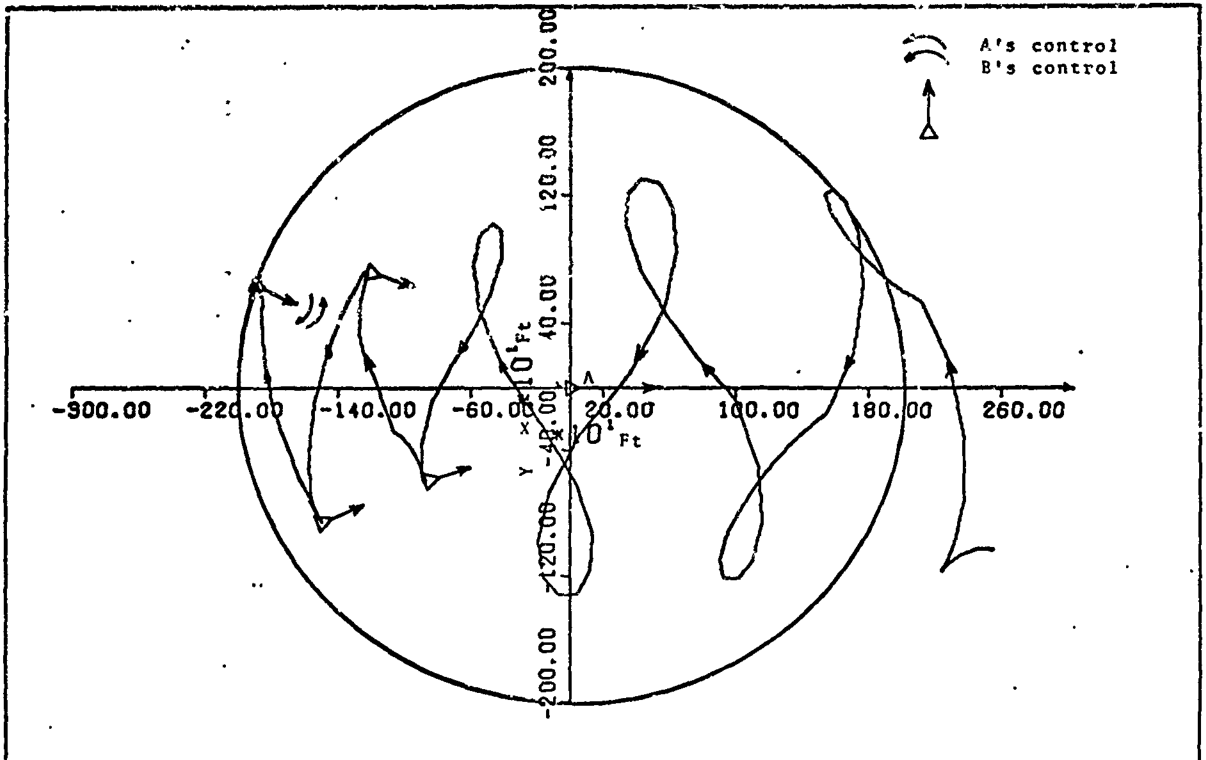

Fig. 12. Scissors Trajectory Successful for B

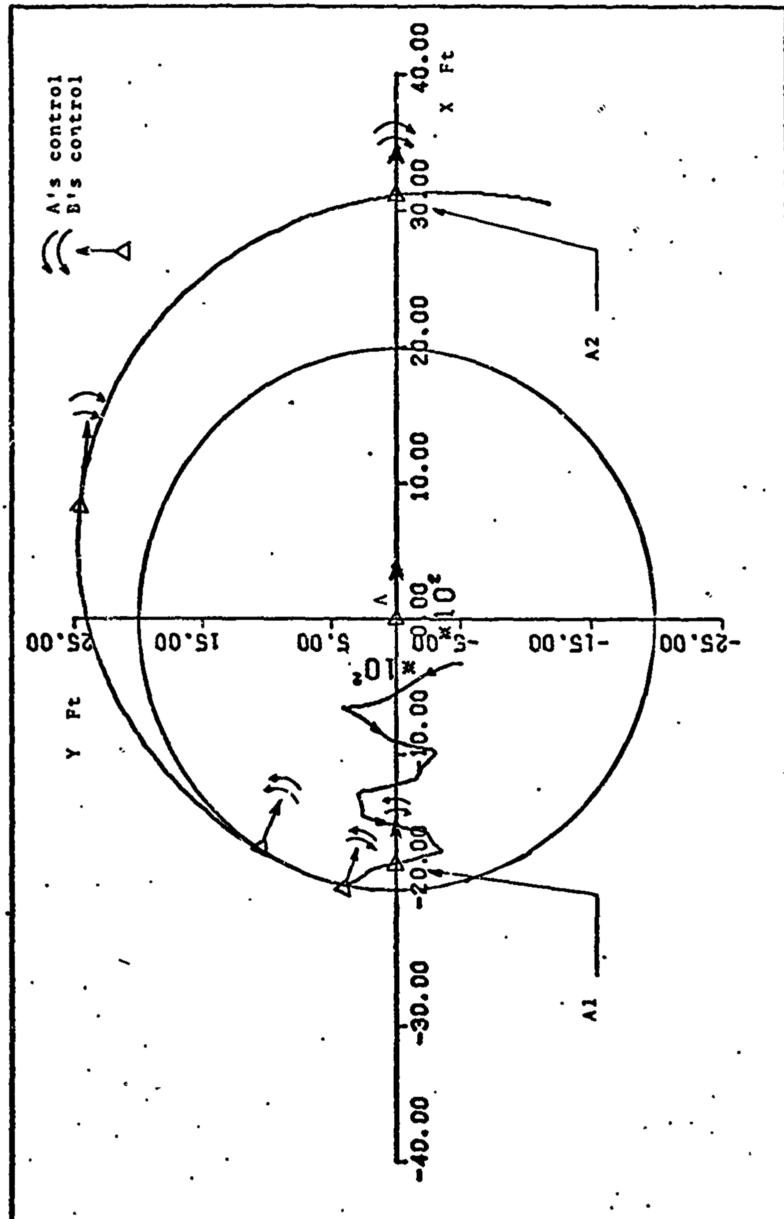

Fig. 13. Dispersal Points on Trajectories Successful for B

Thus the velocity ratio $v = V_A/V_B = 1.1$, and the control ratio $U_{BMAX}/U_{AMAX} = 1.2$. The weapon range is $L = 2000$ feet, and each aircraft is assumed to need boresight to within approximately $\pm$ 10° at an angle off of within approximately $\pm$ 40°; this gives

$$\phi_A = 0.02 \quad a = 0.2 \quad b = 1.8$$

defining A's success, and

$$\phi_B = 1.98 \quad a = 1.8 \quad b = 0.2$$

defining B's success.

The general pattern of trajectories using these game parameters is shown in Figs. 9, 10, 11, 12 and 13. All of these trajectories terminate on the upper half $((z_f - \theta_f) > 0, \pi)$ of an isocost, and because of inverse symmetry about $z_f - \theta_f = 0, \pi$, they are sufficient to show the behavior of trajectories terminating on the lower half. A lag ending for A is considered as $z_f > \theta_f > 0$ or $z_f < \theta_f < 0$. A lead ending for A is considered as $z_f < \theta_f > 0$ or $z_f > \theta_f < 0$. Note also that $\theta_f = 0$ is the boresight condition for A (Fig. 1, page 8).

Application of Surface Solutions

Analysis of the trajectories in Fig. 9, shows that the A singular arc is joined to the TS by one further arc. Consequently, the closed form surface technique is applied to locate the exact point at which the singular arc ends, and the point on the TS at which the singular trajectories terminate. This is repeated for a number of costs, producing curves that represent the end of singular arcs in the game

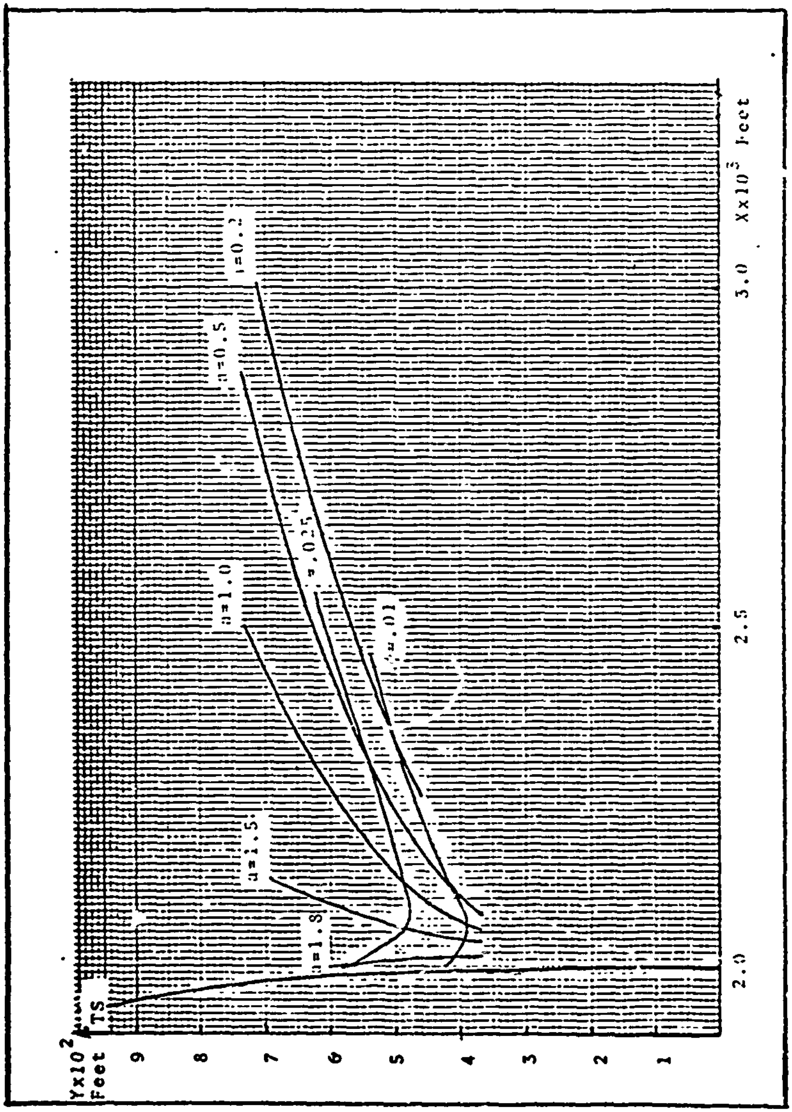

Fig. 14. Variation in the End Points of A Singular Arcs

space (Fig. 14, page 44), and the termination of singular trajectories on the TS (Fig. 15, page 46). Curves are generated for various weighting factors.

<u>End of Singular Arcs</u>. Fig. 14 shows that if A's success is strongly oriented to boresight (small "a" and low θ_f range), then A turns out of his straight dash into B earlier (and farther away) than he would for termination requirements of a small aspect angle well off-boresight (large "a" and low $(z_f - \theta_f)$ range). Also, as the cost decreases to zero, there appears to be a lower limit on y. This is found by using Eq (4-29) for an A singular surface. All the singular curves must lie on this surface, so that for $\sigma_B > 0$

$$y_{min} = (v - 1)\, R_B \text{ and } z = 0 \qquad (5\text{-}1)$$

For the parameters used, y_{min} is 400 feet and this agrees closely with Fig. 14. This indicates that B has to be at least that distance off to one side for A to go singular.

<u>Termination of Singular Trajectories</u>. The general pattern of trajectories in Fig. 9, (page 38) suggests that a large number of trajectories result in a singular arc, which itself terminates, via one more arc, at one point on the TS. Thus the terminal curves of Fig. 15 are lines on the TS where termination occurs from a large number of starting positions. The lines extend over a very much lower range of θ_f than the total range of θ_f possible for a given cost. For example, if a = 0.2 and ϕ = 0.02, then from Eq (3-3) $|\theta_f| \leq 12.6°$, whereas Fig. 15 shows that the singular

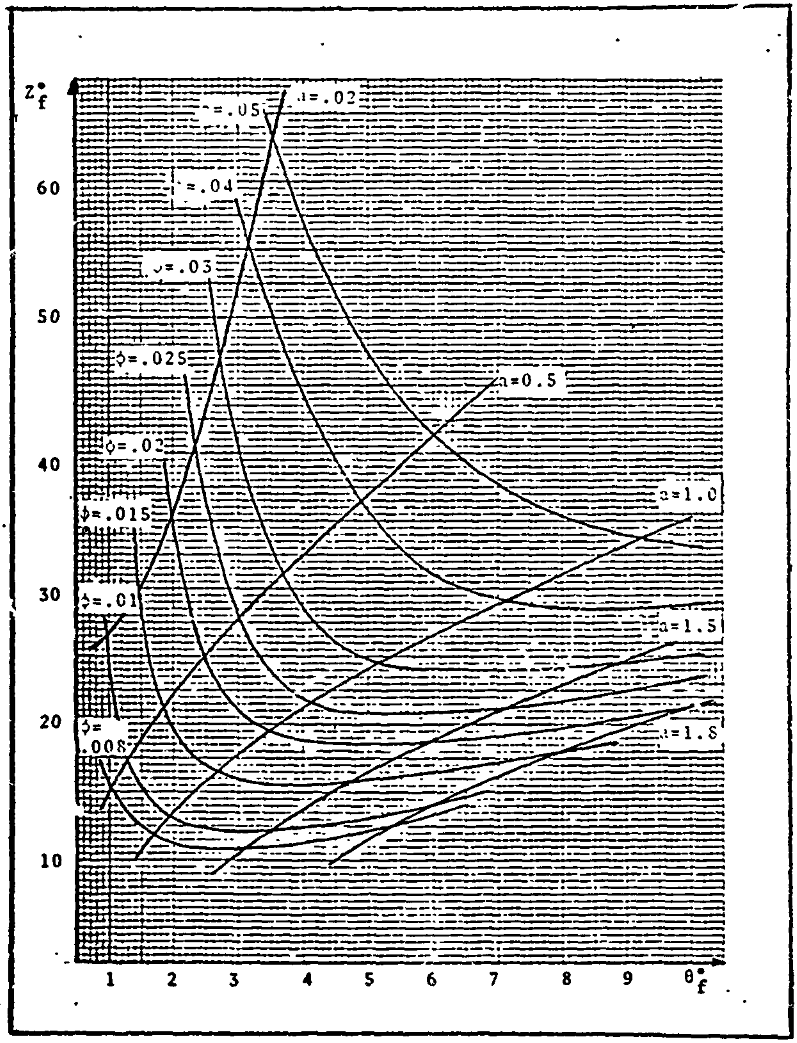

Fig. 15. Variation in the Termination Points of A Singular Trajectories

trajectory terminates at $\theta_f \gtrsim 2.0°$. However proximity to boresight is gained by A at a cost in angle off (z_f) which is high; indeed for low values of "a", the singular curve gives values of z_f close to the maximum for a given isocost. Thus Fig. 15 can be used to select weighting factors and costs to represent the angular requirements, at a given range, of differing weapon systems. The drawback in using these trajectories is that they may be globally non-optimal for the solution in the large; this requires further consideration of the game, and is dealt with in Chapter VI.

Application of Trajectory Analysis

Before attempting to parameterize, trajectory projections are analyzed for the existence, and general position of switching surfaces, singular arcs, and dispersal surfaces. As the scissors trajectories illustrate, points on switching surfaces exist in large numbers, and considerably complicate trajectory analysis.

B Singular Arc. As shown on page 45 a singular arc for A (and its symmetrical counterpart) is found for trajectories that terminate with some lag for A ($(z_f - \theta_f) > 0$). A singular arc was also found for B, as shown in Fig. 10, page 39). However as this arc is connected to the TS via two other arcs ($(\sigma_A < 0, \sigma_B > 0)$, $(\sigma_A > 0, \sigma_B > 0)$), a B singular closed form solution has not been attempted. An interesting observation is that the arc is degenerate in the sense that nonsingular arcs joining it on one side are almost

parallel in the xy projection (Fig. 10). The B singular arc thus resembles what Isaacs calls a semi-universal surface (Ref 8:196). This can be shown analytically by considering a singular and a nonsingular arc backwards from a point $\underline{X}_B$ ($t_f = 0$) on the singular arc. The singular arc state solutions given by Eqs (4-7) to (4-9) are subtracted from the nonsingular arc state solutions of Eqs (4-1) to (4-3) giving the following differences in states at a time t backwards along each arc from $\underline{X}_B$

$$\Delta x(t) = 2R_B \sin \frac{\sigma_B t}{2} \cos \left(\frac{\sigma_B t}{2} - (z_B - \sigma_A t)\right) - R_A \sin \sigma_A t - V_B t \cos (z_B - \sigma_A t) \qquad (5\text{-}2)$$

$$\Delta y(t) = 2R_B \sin \frac{\sigma_B t}{2} \sin \left(\frac{\sigma_B t}{2} + (z_B - \sigma_A)t\right) - V_B t \sin (z_B - \sigma_A t) \qquad (5\text{-}3)$$

$$\Delta z(t) = \sigma_B t \qquad (5\text{-}4)$$

The possible nonsingular-arcs are $\sigma_B = \pm\ 0.24$ and $\sigma_A = -0.2$; so that for $t < 3$ secs, $\sigma_B t/2$ is small and Eqs (5-2) and (5-3) can be approximated by

$$\Delta x(t) \approx (R_B \sigma_B t - V_B t) \cos (z_B - \sigma_A t) - R_A \sin \sigma_A t \qquad (5\text{-}5)$$

$$\Delta y(t) \approx (R_B \sigma_B t - V_B t) \sin (z_B - \sigma_A t) \qquad (5\text{-}6)$$

Substituting $\sigma_B > 0$ gives

$$\Delta x(t) \approx -R_A \sin \sigma_A t \text{ where } \sigma_A < 0 \qquad (5\text{-}7)$$

$$\Delta y(t) \approx 0 \qquad (5\text{-}8)$$

Thus for low values of τ, there is only a small difference in the x and y co-ordinates between the B singular arc and the nonsingular arc joining it with positive B control. This confirms the trajectory plots of Fig. 10. However, as for the A singular case, there still remains the question of global optimality for these trajectories.

Dispersal Surfaces. Direct closed form solutions for dispersal surfaces are not available, but analysis of trajectory projections from the limiting isocost indicates some possible points on a dispersal curve. The locations of some DS points are shown in Fig. 9 and 13; these are special DS points representing a choice of control for both A and B. Considering the points in two groups:

1. A_1 and B_1. The points A_1 and B_1 are similar because they probably both lie on the closest DS to the terminal surface, and are thus likely to lie on the bounds of a closed region of the game space (Ref 6:31). (For a two dimensional problem this is definitely so; for a three dimensional problem, closure does not necessarily occur). Inside these regions (for point B_1) success for A is inevitable for optimal play, regardless of B's actions. Thus point B_1 appears to be the closest that B can reasonably allow A to approach, in a direct tail chase situation, before turning. If B delays, then A's success is inevitable, and B's condition at capture becomes worse the longer he waits

to turn. A similar situation occurs at A_1 for A with respect to B's final success.

$\underline{A_2 \text{ and } B_2}$. With the A_2, B_2 dispersal points, it is likely that they do not lie in the first DS that the generating trajectories intersect. So for a straightforward game of pursuit and evasion with no role change, these points are likely to be in part of the game space which results in a draw in optimal play.

The location of these points and the DS points on the TS (page 34) will enable dispersal curves to be located for a given isocost; either by parameterisation of many trajectories, or by using a numerical search technique, based on the DS conditions on page 29 , to extrapolate from a known DS point.

Summary

In this chapter, the closed form solutions are applied in an attempt to solve the problem completely. Two techniques are tried: direct application of the closed form surface solutions, and analysis of trajectories. Some general trajectory characteristics are observed, in particular, the correspondence of many trajectories to a scissors maneuver. Precise location of A singular arcs enables some criteria to be set for selecting the payoff parameters to match various weapon capabilities. A singular B arc is found, and an interesting approximation to a semi-universal surface is confirmed. Finally, some points on dispersal surfaces are located, and two methods for locating the surfaces are suggested.

Success in fully defining the game space is limited, largely by time. It does however make a start and lead the way to a complete solution of the game.

VI. The Role Problem

Until the game is completely solved, analysis of the role problem is limited. However, in this chapter, a method of role determination is discussed. Capture regions can be determined for the game, and a method is proposed of estimating regions of advantage which assist in role determination outside capture regions. The results of Chapter V are related to this concept, and finally the relationship with the classical game is discussed.

Role Determination

The role determination problem is considered to consist of two parts:

1. Which player has an advantage, and to what extent?
2. Based on the relative advantage, what is now the objective (the role decision) of each player?

Both parts are inter-related because the degree of advantage obviously depends on the objectives. From the pilots point of view, the questions of role are:

1. How good or bad is my situation?
2. What should be my goal?

For example, if the situation is good, then should the objective be to continue, or to escape while that is possible? This decision is easier if the amount of advantage is high or low. For example, if one pilot is about to shoot down an opponent, he is unlikely to flee, and his opponent has little choice but to vigorously avoid defeat. However, if neither has a distinct

advantage, then each may remain, or may be able to escape the combat completely.

The choice of objectives in the role decision problem also depends on the strategy needed to achieve it. For example, it may be necessary to risk a close shave with an opponent in order to reach a position of advantage. The problem of role is thus very complex, and casts doubts on the validity of a zero-sum game. However, a zero sum game is easier to handle, and gives some solutions to the role problem.

In principle, for the game as formulated, if the initial states are known, the outcome is also known (assuming optimal play by both sides). Thus the game space (G) can be divided into four distinct regions resulting in the four possible outcomes (page 11). These regions are: R_{SA} (success for A), R_{SB} (success for B), R_{SAB} (mutual kill), and R (draw).

Consider a problem, such as this one with the Lynch pay-off, in which a mutual kill, R_{SAB}, is effectively nonexistent; if the states are in R_{SA}, and A plays optimally, then the states remain in R_{SA} for the duration of the game, and termination occurs successfully for A ($\phi < \phi_A$ in Fig. 16, page 54). A similar argument applies to B. In the other region R, both players must play optimally for the states to remain in this region and a draw outcome. If one player does not play optimally, then the states can move closer to a region more favorable to his opponent. The boundaries between the regions constitute "barriers" in the sense that they are not crossed in optimal play. (Appendix A, page 74).

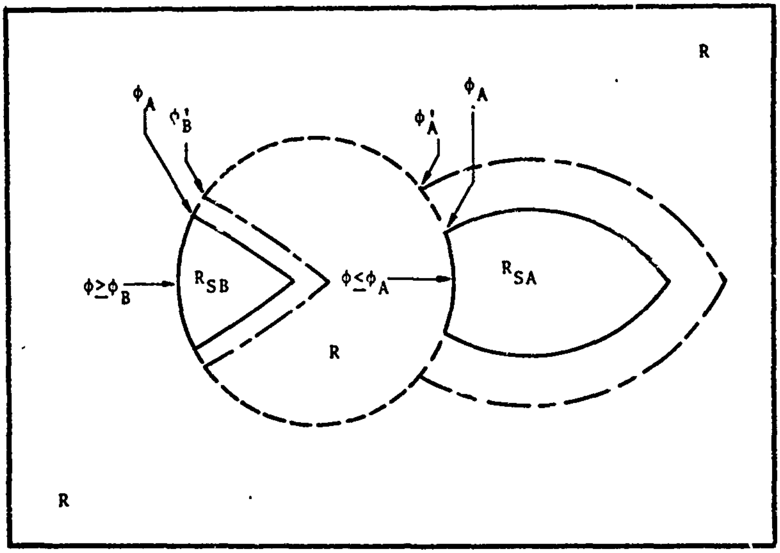

Fig. 16 A 2-D Representation of Regions of Success, and Regions of Advantage Based on Optimal Play.

A method of role determination now exists for regions R_{SA} and R_{SB}. If $\underline{x} \in R_{SA}$, this implies A has an almost total advantage and should press home his attack (i.e. a definite pursuer); conversely, B has little hope of avoiding A and should concentrate entirely on evasion (i.e. a definite evader). A similar argument applies for $\underline{x} \in R_{SB}$.

The situation in the region R, however, is not so clear. Neither has a decisive advantage, nor is under an immediate threat. Each player presumably wants to maneuver his opponent into his own success region, though conceivably each may wish

to escape while the going is good. The division of G may now be represented by

$\underline{x} \in R_{SA} \rightarrow$ role determined/A's success (capture) region (6-1)

$\underline{x} \in R_{SB} \rightarrow$ role determined/B's success (capture) region (6-2)

$\underline{x} \in R \rightarrow$ role uncertain/draw occurs for optimal play (6-3)

where $G = R + R_{SA} + R_{SB}$

R_{SA} and R_{SB} are capture regions resulting from this zero sum pursuit-evasion game in which A is the pursuer, B the evader, and vice-versa. Thus this game formulation is useful in defining regions of extreme advantage and disadvantage (indeed complete success under optimal play by the pursuer). These regions may be very small if the combatants have a similar performance (Ref 6:32), and consequently, it is important to extend the role problem into the region R. The apparent choice of objectives in R suggests a non-zero sum game; however, the problem here is whether the zero sum game and payoff, as formulated, can be used in this region to determine role.

Regions of Advantage

The first part of the role problem in R is to determine regions of advantage.

One method of producing these regions is to expand the solutions outwards from the successful limits, by changing the terminal costs to ϕ_A' and ϕ_B' ($\phi_A' > \phi_A$ and $\phi_B' < \phi_B$). As shown in Fig. 16 (page 51), this would result in expanding around

the success regions R_{SA} and R_{SB}. These expanded regions indicate states which would result in a draw under optimal play for the game as formulated. The term draw is misleading, since it may include a wide range of terminal states: from a close shave ($\phi \approx \phi_A$), to a head on (ϕ = neutral outcome, ϕ_N). However, if $\phi_A < \phi < \phi_A'$ is considered a good draw outcome for A, and $\phi_B' < \phi < \phi_B$ is considered a good draw outcome for B, then the regions that result in these outcomes may be considered as regions of advantage.

The drawback with this method is that an indecisive outcome results, and the game continues indefinitely. Also, the assumption that a draw outcome close to A's success area is good for A and places him at an advantage, is not necessarily true. Indeed, it may be that if A just misses B, then A is at a grave disadvantage (for example, on an overshoot). This method is useful however, because the regions of advantage generated represent the extent to which an opponent could avoid a kill, should he wish to, by playing optimally.

Another method avoids the draw outcome, but uses trajectories which are non-optimal in the strict sense of the game as formulated.

Consider that in a region of advantage to A, his objective is to minimize ϕ, and B's objective is to allow A to come very close to success, in the hope that by doing so, B will move into his own region of advantage. In other words B is deliberately allowing himself to be drawn close to A's

capture regions, R_{SA} (this may also occur if B wants a draw, but keeps making mistakes). This strategy is non-optimal in the game as formulated; indeed B is virtually playing a non-zero game by co-operating (at least for a time) with A's objective. Consequently, trajectories resulting in R_{SA} (or extending from R_{SA} in a backwards solution) are non-optimal for B; nevertheless, they do represent almost the best that A can achieve, with B co-operating or making mistakes. Thus these trajectories may be used to indicate general regions of advantage to A. Because player B is, at least temporarily, helping his opponent, the resulting regions of advantage are expected to be large.

A region of advantage for A may now be defined by those starting points in R for which A is successful, if B, either cannot avoid A's successful outcome, or is using a strategy which would result in A's success (conversely for B). This gives

$$\underline{x} \in R_{AA} \text{ iff B plays to } \phi < \phi_A \qquad (6\text{-}4)$$

$$\underline{x} \in R_{AB} \text{ iff A plays to } \phi > \phi_B \qquad (6\text{-}5)$$

where R_{AA} is A's region of advantage, and R_{AB} is B's region of advantage. The possibility also exists that the regions of advantage may intersect $(R_{AA} \cap R_{AB} \neq 0)$.

The first part of the role determination in R problem is now

$$\underline{x} \in R_{AA} \rightarrow \text{A's advantage} \qquad (6\text{-}6)$$

$$\underline{x} \in R_{AB} \rightarrow \text{B's advantage} \qquad (6\text{-}7)$$

$$\underline{x} \in (R_{AA} \cap R_{AB}) \rightarrow \text{Advantage to both} \qquad (6\text{-}8)$$

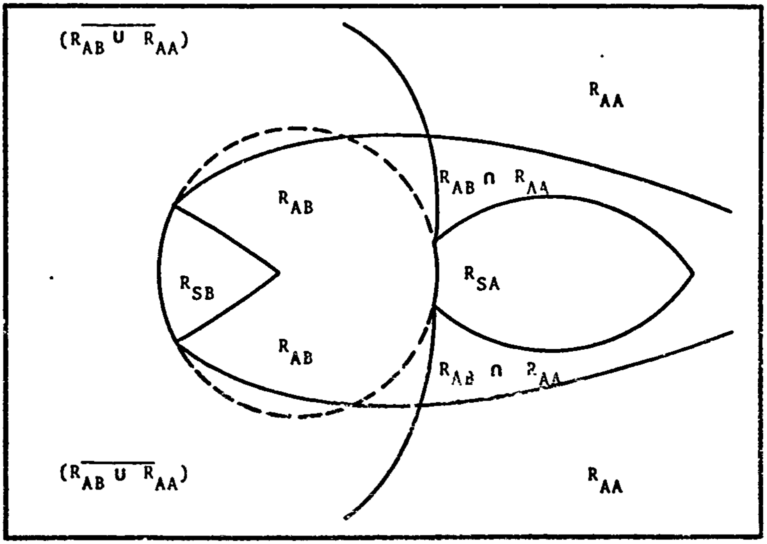

Fig. 17. A 2-D Representation of Regions of Advantage Based on Non-optimal Play.

and is shown in Fig. 17, page 58 . It follows that

$$R_{SA} \subset R_{AA}$$

$$R_{SB} \subset R_{AB}$$

To be complete, if R_{AA} and R_{AB} do not fill R ($R_{AA} \cup R_{AB} \neq R$), then $\underline{x} \in (\overline{R_{AA} \cup R_{AB}}) \rightarrow$ no advantage to either.

The second part of the role determination is to decide what the objective should be. For example, if B is in A's region of advantage, R_{AA}, should he play optimally for the temporary respite of a draw, or should he allow himself to get close to A's success region, R_{SA}, in order to put himself in a

good position. Conversely, should A continue to make the most of his advantage and play towards a kill. Both may decide, if able, to escape. Clearly, this problem requires other game formulations.

Regions of Advantage and the Dispersal Surface. The game formulation gives trajectories and surfaces backwards from an isocost ϕ_A on the TS, hence regions of success and advantage (as defined above) should emerge. Looking at the game in this way, dispersal surfaces do not terminate trajectories, although they still are used to enclose regions of success and advantage. Consider a 2-D example shown in Fig. 6 (page 29), with a DS point X_{DS} on a trajectory leaving the TS from ϕ_A. Continue along the trajectory from X_{DS} to Y. For the game as formulated the path YX_{DS} is strictly non-optimal because YZ results in a better value (ϕ_2) for B. Thus point Y lies in the region R, and X_{DS} defines a point on the boundary of A's success/capture region, R_{SA}. However, if B accepts the outcome $\phi < \phi_A$, then YX may be used to indicate A's regions of advantage R_{AA}.

Results and Role Determination

As suggested in Chapter V, it is likely that DS points A_1 and B_1 of Figs. 9 and 13 are on the surface that encloses the success regions R_{SA} and R_{SB}. Thus, as expected, the capture regions themselves are relatively small (Ref 6:46). Trajectories extending beyond these points show a general pattern; virtually all of B's successful trajectories are

forward of him, are within his weapon range on either side, and are overshoot scissors trajectories. This general pattern is a reasonably valid result for a slower more maneuverable machine. On the other hand, A's successful trajectories cover a larger part of the x-v plane, are not so often scissors, and include singular "catch-up" arcs. Here again, the general pattern is consistent with the faster aircraft. Furthermore, the trajectories show that A has a larger region of advantage than B. Intersection of R_{AA} and R_{AB} also seems likely to occur forward of A's terminal area.

Consider again the DS point B_2 (Fig. 9, page 38); if B decides to avoid, one of the ways he could do so would be to remain on a collision course and accept a neutral ϕ (i.e. play optimally in the hope that A will do likewise). However, if he decides this is not good enough, and tries to maneuver into an attacking position, he may first be obliged to accept a close shave from A by entering, and holding a turn along a trajectory terminating in ϕ_A. Then A, having missed (hopefully) and overshot, gives B his opportunity. Whether this maneuver is the best for B is uncertain, what can be said however, is that as B_2 lies in the region R, and on a ϕ_A trajectory, it is a point in the surface of A's region of advantage R_{AA}. The indication at B_2 is that that point is in R_{AA} for all angles off, z.

A different situation arises at point B_3 (Fig. 9, page 38) here B is overshooting A and is turning outwards. This seems

contrary to experience because he is so close to the TS that by turning in, it appears he could easily achieve a draw outcome of $\phi > \phi_A$, and get into an advantageous position. The implication is that this part of the trajectory lies in R, and indeed on the surface of R_{AA}. However, here the region of B's advantage R_{AB}, extending forward from B's success region, is very close. Thus B, by turning in at B_3 is likely to move into R_{AB} a lot quicker, and certainly with less danger, than following the trajectory that results in ϕ_A, and accepting a close shave.

One more example further illustrates that, although ϕ_A trajectories in R are useful to locate regions of advantage, they are not necessarily the best strategies for either of the players. Their strategies will result from the role decision problem. Consider a scissors maneuver in which A is overshooting (Fig. 13 page 42), it is likely that most of the trajectory is in R, and defines R_{AB}; however, as such, A is not obliged to accept the scissors as being the best way of regaining the advantage (although he may be so obliged because of limited information etc). Indeed some temporary draw outcome (created by say, a singular dash) could possibly get A into his region of advantage R_{AA} faster, and in less danger, than continuing to B's success region R_{SB}, and finally ϕ_B. This trajectory is a "degenerate scissors" in the sense that B's movement about A's longitudinal axis is small. Further investigation may reveal a singular arc here. Thus the

results show that:

1. Using trajectories beyond capture regions is useful in establishing regions of advantage on which role decisions can be based.
2. These trajectories are not necessarily the best way to move from one region of advantage to another.

A full solution to the game is needed to confirm the prospect in subpara 1. and throw further light on the problem in subpara 2.

Relation to the Classical Game

It is interesting to relate the results of this game to the classic game formulation and resulting barriers (Ref 8). Using Lynch's work on barrier closure (Ref 9:106) it was found that, with the parameters used in Chapter V for detailed analysis, barrier closure did not occur. Thus B has no chance of escape (for $0 < tf < \infty$, and capture defined as B simply crossing the TS), and conversely A can always do so (at least when outside the TS).

Thus the general pattern of trajectories and areas of advantage found using the Lynch payoff without time, agrees with the classical formulation. However, the introduction of a payoff that reflects a desire to remain in combat rather than escape appears to have created capture regions for both A and B and altogether made the model richer and more realistic.

Summary

The game of kind formulation from a terminal payoff

can be used to determine capture regions and to indicate regions of advantage. For the capture regions, role is determined. The regions of advantage are useful for making role decisions, but do not indicate the actual strategy to use. Results so far confirm this, but the complete solution to the game is required. The classic game gives the same basic results in a degenerate form.

VII. Conclusions and Recommendations

Conclusions

The primary objective of this thesis was to study the application and validity of the generalized Lynch payoff to the problem of role determination in a free time, zero sum, perfect information differential game model of a two aircraft combat. The secondary objective was to establish techniques for using the payoff in simple dynamic models.

A constant speed, horizontally planar dynamic model was used, and time was dropped from Lynch's payoff. This resulted in valid controls at the end of the game, and closed form solutions were derived for the terminal co-states, for the states and co-states along constant control arcs, and for switching and singular surfaces at junctions. These solutions were then used directly and indirectly to partition the game space outward from the terminal surface. This was not fully achieved, though isolated singular arcs were identified and confirmed for this type of game, and their location and termination in relation to various payoffs was found. Some dispersal points were located, which gave an indication of partitioning and capture regions for each combatant. A strong boost to the validity of this approach are trajectories that correspond to practical scissors maneuvers. Unfortunately, much of the length of these trajectories are likely to be non-optimal for the game as formulated; however, a technique

was proposed, using these trajectories and their intersecting surfaces, to estimate regions of advantage for each player. A comparison with Isaacs' classical barrier formulation agrees broadly with the general pattern of trajectories produced by this formulation.

Hence the thesis has achieved both objectives to some extent, and enables a limited main conclusion that the Lynch payoff, in its terminal form, shows some promise for producing solutions for simple dynamic models that bear some resemblance to actual combat. Furthermore, the payoff can handle a wide range of weapon systems.

Recommendations

This thesis should definitely not be regarded as approaching a reasonably valid differential game model of aerial combat. Not only must the solution in the large be fully completed, far too many important factors have had to be assumed away, in order to make some initial progress.

Consequently, there are four main recommendations for further investigation:

1. The complete partitioning of the game space and full investigation of the use of advantage regions in role determination. This will involve refining techniques for the location of the game surfaces.
2. The effect of including time in the payoff and the interaction with other game formulations, in

particular a non-zero sum formulation.

3. A full parameter investigation of the solution behaviour.

4. The extension of the game model to include factors such as restricted information, and the extension of the dynamic model to include variable velocity and altitude.

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Appendix A

Differential Game Theory

Purpose

The purpose of this appendix is to summarize differential game theory applicable to the two person, zero sum, perfect information game considered in this study. References 1, 5 and 8 form the basis of this summary.

Game Formulation

Two players are constrained by their dynamics to

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}, \underline{v}, t), \; \underline{x}(t_o) = \underline{x}_o \qquad \text{(A-1)}$$

where $\underline{x}$ is the n - dimensional state vector, $\underline{u}$ is the m - dimensional control vector of one player, and $\underline{v}$ is the p - dimensional control vector of the other. For simplicity, consider $\underline{u}$ and $\underline{v}$ each to be one-dimensional. Both u and v may be subject to constraints. The players may also have to satisfy a terminal constraint.

$$\psi(\underline{x}\,(tf), tf) = 0 \qquad \text{(A-2)}$$

A cost function (payoff) is defined as

$$J = \phi(\underline{x}\,(tf), tf) + \int_{to}^{tf} L\,(\underline{x}, \underline{u}, \underline{v}, t)\, dt \qquad \text{(A-3)}$$

and the aim is to find the controls u* and v* such that

$$J\,(u^*, v) \leq J\,(u^*, v^*) \leq J\,(u, v^*) \qquad \text{(A-4)}$$

If u^* and v^* can be found, the pair (u^*, v^*) is called a saddle point of the game, and $J(u^*\ v^*)$ is called the value.

Necessary Conditions for a Solution

The existence of a solution depends on the commutative condition that

$$\min_u \max_v J(u, v) = \max_v \min_u J(u, v) \tag{A-5}$$

A necessary condition for a saddle point is that the Hamiltonian (H) defined as

$$H(\underline{x}, \underline{\lambda}, u, v, t) = \underline{\lambda}^t \underline{f} + L \tag{A-6}$$

must be minimized over the admissable set of u, and maximized over the admissable set of v, such that

$$H^* = \min_u \max_v H = \max_v \min_u H \tag{A-7}$$

giving, if there are no control constraints

$$H_u = 0 \quad \text{and} \quad H_v = 0 \tag{A-8}$$

The commutative condition of Eqs (A-5) and (A-7) is satisfied if H is separable for u and v. If u and v appear linearly in H and are constrained by

$$|u| \leq u_{max} \text{ and } |v| \leq v_{max} \tag{A-9}$$

with switching functions S_u and S_v defined as

$$S_u(\underline{x}, \underline{\lambda}) = H_u \text{ and } S_v(\underline{x}, \underline{\lambda}) = H_v \tag{A-10}$$

The saddle point controls are given by

$$\begin{aligned} u^* &= u_{max} \quad \text{if} \quad S_u < 0 \\ u^* &= -u_{max} \quad \text{if} \quad S_u > 0 \\ v^* &= v_{max} \quad \text{if} \quad S_v > 0 \\ v^* &= -v_{max} \quad \text{if} \quad S_v < 0 \end{aligned} \tag{A-11}$$

Where there are no state dependent control constraints, the n - dimensional co-state vector λ is given by

$$\dot{\underline{\lambda}} = -H_{\underline{x}} \tag{A-12}$$

subject to the transversality conditions

$$\underline{\lambda}(tf) = \phi_{\underline{x}}\ (tf) + \nu\ \underline{\psi}_{\underline{x}}\ (tf) \tag{A-13}$$

$$H(tf) = \phi_t\ (tf) + \nu\ \underline{\psi}_t\ (tf) \tag{A-14}$$

where ν is a constant multiplier.

If time does not appear explicitly in either H or the control constraints, then H is constant. Also, if the problem is one of free final time, then Eq (A-14) implies that

$$H(tf) = H\ (t) = 0 \tag{A-15}$$

If u and v are expressed as functions of time and the initial states, then this is an open loop solution. If they are expressed as functions of time and the instantaneous states, then the solution constitutes a closed loop law.

Singular Controls and Surfaces

When the controls u and v appear linearly in H the possibility of solution arcs with singular controls exists. Using the switching functions defined in Eq (A-10), H may be arranged as

$$H = \underline{\lambda}^T \underline{g} (\underline{x}) + S_u u + S_v v \qquad \text{(A-16)}$$

Taking the case where $S_u = 0$, H then becomes independent of u and thus minimization with respect to u is not possible. Similarily for v, if $S_v = 0$. However, a necessary condition for a solution on a singular arc is

$$S_u (\underline{x}, \underline{\lambda}) = \dot{S}_u (\underline{x}, \underline{\lambda}) = - - S_u^{2q-1} (\underline{x}, \underline{\lambda}) = 0 \qquad \text{(A-17)}$$

where successive differentiation yields a function

$$S_u^{2q} (\underline{x}, \underline{\lambda}, u) = 0 \qquad \text{(A-18)}$$

which is explicit in u and may be solved to yield a candidate for u* on the singular arc. This must satisfy the generalized Legendre-Clebsch condition

$$(-1)^q \frac{\partial}{\partial u} \left(S_u^{2q} \right) \geq 0 \qquad \text{(A-19)}$$

and at junctions with nonsingular arcs, the condition

$$\frac{\partial}{\partial u} \left(S_u^{2q} \right) \Bigg|_{\text{junction time}} \leq 0 \qquad \text{(A-20)}$$

must also be satisfied.

Equation (A-17) gives a singular surface of dimension $2n - 2q$ in $(x\text{-}\lambda)$ space; with the additional constraint of Eq (A-15), in a free time problem the dimension becomes $2n - 2q - 1$. If $n = 3$ and $2 = 1$ this yields a singular surface in the state space on which all singular arcs must lie. Hence intersection with this surface is a necessary, but not sufficient condition for switching to a singular arc. Another important property of singular arcs is that they can be joined by an infinity of paths.

Switching Surface

A switching surface consists of points at which the controls are discontinuous. From Eqs (A-11) and (A-17) the controls switch only when

$$S_u = 0 \text{ (for u), and } S_v = 0 \text{ (for v)} \qquad \text{(A-21)}$$

This condition defines a switching surface in $(x\text{-}\lambda)$ space and implies that a singular surface is a particular type of switching surface, also satisfying Eq (A-17).

The Barriers and Games of Kind and Degree

Isaacs (Ref 8:35) classisfies games into those of "kind", and those of "degree". In the game of kind, the primary interest is whether or not termination (as defined by the game) occurs. For a game of degree however, termination is assumed to occur, and the players' objectives are to hasten or delay termination, or to minimax a

continuous payoff.

There is not a clear distinction between these classes of games (Ref 8:230); a game of kind can be regarded as one of degree in which bounds are placed on the terminal payoff. Termination of the game is then defined as occurring only when the constraints are satisfied and the payoff is within the specified bounds. In the usual classical formulation (Ref 8), the terminal payoff, in a game of degree, is the final time (tf), and the terminal constraint is the circular region $x^2 - y^2 = L^2$. The game becomes a game of kind when the bounds $0 < tf < \infty$ are placed on the payoff, and the only acceptable outcome must be within these bounds. Hence the usual classical formulation of a game considers the possibility of the terminal constraint ever being satisfied. Games of degree with terminal payoffs can, in a similar way, be treated as games of kind (Ref 6).

For a game of kind there may be a surface in the game called a "barrier". In the classical formulation this surface consists of those starting points which end at the limit of all possible terminations (BUP). In the more general sense, a barrier consists of starting points which terminate on the bounds set by the terminal payoff. If the barrier encloses entirely some region of the game space then the space is divided into distinct regions of success and failure (Ref 3:66).

Dispersal Surface

There are many other possible surfaces (Ref 8). One of the most common is the dispersal surface, defined by

$$J_1\ (\underline{x}_{DS}) = J_2\ (\underline{x}_{DS}) = ----J_K\ (\underline{x}_{DS}) \qquad \text{(A-22)}$$

where $J_K\ (\underline{x}_{DS})$ represents the value of the payoff along the K^{th} possible path from a single point $\underline{x}_{DS}$ on the dispersal surface. Figure 6 illustrates this for a two dimensional problem with $K = 2$. Intersection with the dispersal surface thus represents a decision point at which one, or sometimes both players may be able to choose strategies, knowing that in either case the payoff will be the same regardless of the choice. In representing the limit of optimality of a trajectory, it is useful for terminating arcs and bounding regions.

Appendix B

Derivation of Closed Form Solutions

Purpose

The purpose of this appendix is to show the derivation of closed form solutions to the game formulated in Chapter III.

Terminal Co-states λ_{x_f} and λ_{y_f}

The terminal co-states λ_{x_f} and λ_{y_f} are given by

$$\lambda_{x_f} = \Phi_{x_f} \qquad\qquad \lambda_{y_f} = \Phi_{y_f}$$

where
$$\Phi = \frac{(a+b)}{2} - (a \cos (z_f - \theta_f) + b \cos \theta_f) + \nu(x_f^2 + y_f^2 - L^2)$$

or
$$\Phi = \frac{(a+b)}{2} - (a \cos (z_f - \theta_f) + b \cos \theta_f) + \nu(r_f^2 - L^2) \tag{B-1}$$

To avoid expressing Φ in cartesian co-ordinates and taking partial derivatives directly, the chain rule can be used giving

$$\Phi_\theta = \Phi_x x_\theta + \Phi_y y_\theta$$

or
$$\lambda_{\theta_f} = L(\lambda_{y_f} \cos \theta_f - \lambda_{x_f} \sin \theta_f) \tag{B-2}$$

and
$$\lambda_{x_f} = (\lambda_{y_f} \cot \theta_f - \frac{\lambda_{\theta_f}}{L} \operatorname{cosec} \theta_f) \tag{B-3}$$

The $\min_A \max_B H(tf) = 0$ condition gives

$$\lambda_{x_f} [V_B \cos z_f - V_A + \sigma_A y_f] +$$

$$\lambda_{y_f} [V_B \sin z_f - \sigma_A x_f] + \lambda_{z_f} [\sigma_B - \sigma_A] = 0 \qquad \text{(B-4)}$$

Substituting

$$\lambda_{\theta_f} = \Phi_{\theta_f} = \frac{1}{2} (b \sin \theta_f - a \sin (z_f - \theta_f))$$

and $\lambda_{z_f} = \Phi_{z_f} = \frac{1}{2} a \sin (z_f - \theta_f)$ (B-5)

and Eq (B-3) into Eq (B-4) gives an equation in terms of λ_{y_f} and the final states which can be solved to give

$$\lambda_{y_f} = \frac{U+W}{2V} \qquad \text{(B-6)}$$

where $U = \sin \theta_f (b \sin \theta_f \sigma_A - a \sin (z_f - \theta_f) \sigma_B)$

$$W = (b \sin \theta_f - a \sin (z_f-\theta_f)) \frac{(V_B \cos z_f - V_A)}{L}$$

$$V = (V_B \cos (z_f-\theta_f) - V_A \cos \theta_f)$$

Substitution for λ_{θ_f} in Eq (B-3) now gives λ_{x_f} in terms of λ_{y_f} and the final states as

$$\lambda_{x_f} = \frac{[2L \lambda_{y_f} \cos \theta_f - (b \sin \theta_f - a \sin (z_f-\theta_f))]}{2L \sin \theta_f} \qquad \text{(B-7)}$$

<u>State and Co-state Solutions</u>

Over an arc of constant controls σ_A and σ_B, the state and co-state equations in cartesian co-ordinates are

$$\dot{x} = V_B \cos z - V_A + \sigma_A y \quad \text{with} \quad x(tf) = x_f$$

$$\dot{y} = V_B \sin z - \sigma_A x \quad \text{with} \quad y(tf) = y_f \qquad \text{(B-8)}$$

$$\dot{z} = \sigma_B - \sigma_A \quad \text{with} \quad z(tf) = z_f$$

and

$$\dot{\lambda_x} = \sigma_A y \quad \text{with} \quad \lambda_x(tf) = \lambda_{x_f}$$

$$\dot{\lambda_y} = -\sigma_A x \quad \text{with} \quad \lambda_y(tf) = \lambda_{y_f} \qquad \text{(B-9)}$$

$$\dot{\lambda_z} = V_B(\lambda_x \sin z - \lambda_y \cos z) \quad \text{with} \quad \lambda_z(tf) = \lambda_{z_f}$$

Nonsingular Solutions. For nonsingular arcs with $\sigma_A \neq 0$ and $\sigma_B \neq 0$, three solutions are obtained immediately as

$$z(t) = (\sigma_B - \sigma_A)\ (t - t_f) + z_f \qquad \text{(B-10)}$$

$$\lambda_x(t) = A' \sin \sigma_A t + B' \cos \sigma_A t \qquad \text{(B-11)}$$

$$\lambda_y(t) = -B' \sin \sigma_A t + A' \cos \sigma_A t \qquad \text{(B-12)}$$

where

$$A' = \lambda_{x_f} \sin \sigma_A tf + \lambda_{y_f} \cos \sigma_A tf$$

and

$$B' = \lambda_{x_f} \cos \sigma_A tf - \lambda_{y_f} \sin \sigma_A tf$$

Letting $tf = 0$, so that along an arc $t < 0$, Eqs (B-10) to (B-12) become

$$z(t) = (\sigma_B - \sigma_A)\ t + z_f \qquad \text{(B-13)}$$

$$\lambda_x(t) = \lambda_{y_f} \sin \sigma_A t + \lambda_{x_f} \cos \sigma_A t \qquad \text{(B-14)}$$

$$\lambda_y(t) = -\lambda_{x_f} \sin \sigma_A t + \lambda_{y_f} \cos \sigma_A t \qquad \text{(B-15)}$$

Substituting in Eq (B-9) for z, λ_x and λ_y gives

$$\dot{\lambda}_z = V_B[\lambda_{x_f} \sin(\sigma_B t + z_f) - \lambda_{y_f} \cos(\sigma_B t + z_f)] \quad \text{(B-16)}$$

and solving with $\lambda_z(tf = 0) = \lambda_{z_f}$ gives

$$\lambda_z(t) = \lambda_{z_f} + R_B[\lambda_{y_f} \sin z_f + \lambda_{x_f} \cos z_f] -$$
$$R_B[\lambda_{y_f} \sin(\sigma_B t + z_f) + \lambda_{x_f} \cos(\sigma_B t + z_f)]$$

(B-17)

or

$$\lambda_z(t) = \lambda_{z_f} - 2 R_B \sin \frac{\sigma_B t}{2} [\lambda_{y_f} \cos(\frac{\sigma_B t}{2} + z_f) +$$
$$\lambda_{x_f} \sin(\frac{\sigma_B t}{2} + z_f)]$$

where $R_B = V_B/\sigma_B$

Finally, for the states $x(t)$ and $y(t)$, a second order DE can be formed from Eq (B-8) giving

$$\ddot{x} + \sigma_A^2 x = V_B(2 \sigma_A - \sigma_B) \sin z$$

Substituting in the solution for $z(t)$ gives

$$\ddot{x} + \sigma_A^2 x = V_B(2 \sigma_A - \sigma_B) \sin[(\sigma_B - \sigma_A) t + z_f]$$

with $x(tf = 0) = x_f$

The solution is now straightforward, yielding

$$x(t) = x_f \cos \sigma_A t + y_f \sin \sigma_A t + R_B [\sin(\sigma_A t - z_f) +$$

(B-18)

$$\sin((\sigma_B - \sigma_A) t + z_f)] - R_A \sin \sigma_A t$$

Differentiation, and substitution for $\dot{x}$ in Eq (B-18) yields

$$y(t) = -x_f \sin \sigma_A t + y_f \cos \sigma_A t + R_B [\cos (\sigma_A t - z_f) - \cos ((\sigma_B - \sigma_A) t + z_f)] + R_A (1 - \cos \sigma_A t) \tag{B-19}$$

Thus Eqs (B-18), (B-19), (B-13), (B-14), (B-15) and (B-16) are the nonsingular arc closed form solutions with tf = 0.

Singular Solutions. Because R_A and R_B are infinite for singular controls, the solutions for nonsingular arcs cannot be applied directly. For an A singular arc, substitution for $\sigma_A = 0$ in Eqs (B-13), (B-14), (B-15) and (B-17), and application of the singular conditions $\lambda_y = 0$ yields

$$z(t) = z_f + \sigma_A t \tag{B-20}$$

$$\lambda_x(t) = \lambda_{x_f} \tag{B-21}$$

$$\lambda_y(t) = \lambda_{y_f} = 0 \tag{B-22}$$

$$\lambda_z(t) = \lambda_{z_f} - 2 R_B \lambda_{x_f} \sin \frac{\sigma_B t}{2} \sin (\frac{\sigma_B t}{2} + z_f) \tag{B-23}$$

Solving the equations of motion directly with $\sigma_A = 0$ gives

$$x(t) = x_f - V_A t + R_B (\sin (z_f + \sigma_B t) - \sin z_f) \tag{B-24}$$

$$y(t) = y_f - R_B (\cos (\sigma_B t + z_f) - \cos z_f) \tag{B-25}$$

For a B singular arc, substitution of $\sigma_B = 0$ into Eq (B-13) with the singular condition $\lambda_z = 0$ gives

$$z(t) = z_f - \sigma_A t \tag{B-26}$$

$$\lambda_z(t) = \lambda_{z_f} = 0 \tag{B-27}$$

and $\lambda_x(t)$ and $\lambda_y(t)$ are still given by Eqs (B-14) and (B-15). Solving the equations of motion directly with $\sigma_B = 0$ gives

$$x(t) = x_f \cos \sigma_A t + (y_f - R_A) \sin \sigma_A t + V_B t \cos (z_f - \sigma_A t) \tag{B-28}$$

$$y(t) = -x_f \sin \sigma_A t + (y_f - R_A) \cos \sigma_A t + V_B t \sin (z_f - \sigma_A t) \tag{B-29}$$

Vita

Peter Jenkins was born on 14 August 1945 in Swansea, Wales. He was educated at Ardwyn Grammar School, Aberystwyth, the Cambridge Grammar School, and Sidney Sussex College, Cambridge, where he graduated in 1967 with a Bachelor of Arts degree in Mechanical Sci[illegible]s.

After attending an App'ied Engineering Course at Royal Air Force College, Cranwell, he served for two and a half years as an aircraft line maintenance engineer at RAF Brize Norton and RAF El Adem. From 1970 to 1972 he participated in the development and introduction into the RAF of its first automated Maintenance Data System. By ancient custom, in 1971 he became an MA.

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This thesis was typed by Ladonna Stitzel.